STUDIA MATHEMATICA
Online First version

Dynamics of weighted composition operators on function spaces defined by local properties

by

Thomas Kalmes (Chemnitz)

Abstract. We study topological transitivity/hypercyclicity and topological (weak) mixing for weighted composition operators on locally convex spaces of scalar-valued functions which are defined by local properties. As main applications of our general approach we characterize these dynamical properties for weighted composition operators on spaces of ultradifferentiable functions, both of Beurling and Roumieu type, and on spaces of zero solutions of elliptic partial differential equations. Special attention is given to eigenspaces of the Laplace operator and of the Cauchy–Riemann operator. Moreover, we show that our abstract approach unifies existing results which characterize hypercyclicity, resp. topological mixing, of (weighted) composition operators on the space of holomorphic functions on a simply connected domain in the complex plane, on the space of smooth functions on an open subset of $\mathbb{R}^d$, as well as results characterizing topological transitiviy of such operators on the space of real analytic functions on an open subset of $\mathbb{R}^d$.

1. Introduction. Dynamical properties like topological transitivity/hypercyclicity and topological (weak) mixing for weighted composition operators on various function spaces have been investigated by many authors in different settings. Recall that an operator, i.e. a continuous linear self-map T on a topological vector space E, is called *topologically transitive*, resp. *topologically mixing*, if for every pair of non-empty, open subsets U, V of E the sets $T^m(U)$ and V intersect for some $m \in \mathbb{N}$, resp. for all sufficiently large $m \in \mathbb{N}$, whereas T is called *topologically weakly mixing* if $T \oplus T$ is topologically transitive on $E \oplus E$. Moreover, T is *hypercyclic* if there is $x \in E$ such that its orbit under T, i.e. the set $\{T^m x; m \in \mathbb{N}_0\}$, is dense in E. Clearly, every hypercyclic operator is topologically transitive, and the converse holds for operators on separable, complete, metrizable

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2010 *Mathematics Subject Classification*: Primary 47A16, 47B33; Secondary 46E10.

Key words and phrases: hypercyclic operator, weighted composition operator, topologically mixing operator, locally convex spaces of functions, ultradifferentiable functions, elliptic differential operator.
Received 9 January 2018; revised 7 June 2018.
Published online 13 May 2019.

DOI: 10.4064/sm180109-8-6

topological vector spaces by Birkhoff's Transitivity Criterion [GEPe, Theorem 2.19].

The most prominent setting for hypercyclic weighted composition operators is the space $\mathscr{H}(X)$ of holomorphic functions endowed with the compact-open topology on an open subset X of the complex plane. Starting from Birkhoff's result [Bi] from which it follows that the translation operators $T_a(f)(\cdot) := f(\cdot + a)$, $a \in \mathbb{C} \setminus \{0\}$, on the space of entire functions $\mathscr{H}(\mathbb{C})$ are hypercyclic, many authors have generalized that result in various directions by considering more general (weighted) composition operators $C_{w,\psi}(f) := w \cdot (f \circ \psi)$ for $f \in \mathscr{H}(X)$, where $w \in \mathscr{H}(X)$ and $\psi : X \to X$ is holomorphic (see e.g. [BeMo], [GEMo], [YoRe], [Bes] and references therein). For holomorphic functions in several variables some partial results on hypercyclicity for special unweighted composition operators have been obtained in [AbZa], [Ber], and [LS], and only recently for arbitrary composition operators (even for holomorphic functions in several variables on a connected Stein manifold) in [Za].

Considering harmonic functions instead of holomorphic functions, in [Dz], [Ar] hypercyclicity of generalized translation operators has been investigated in this context while sufficient conditions for hypercyclicity of such special composition operators on kernels of elliptic partial differential operators have been presented in [CaMu]. A sufficient condition for hypercyclicity of general composition operators on arbitrary kernels of partial differential operators was obtained in [KaNi].

While for unweighted composition operators on the space $\mathscr{A}(X)$ of real analytic functions on an open subset X of $\mathbb{R}^d$ topological transitivity has been investigated in [BoDo], hypercyclicity and topological mixing of weighted composition operators on the space $C^\infty(X)$ of smooth functions, where again $X \subseteq \mathbb{R}^d$ is open, have only very recently been characterized in [Pr].

For Banach spaces of functions, dynamical properties of (weighted) composition operators have also been investigated. There are many results in the context of Banach spaces of holomorphic functions (see e.g. [BoSh], [GaMo], [MiWo] and references therein). Characterizations of hypercyclicity of weighted composition operators on Banach spaces of continuous functions and on $L^p(\mu)$-spaces have been obtained in [Ka].

The aim of this paper is to present a general approach to topological transitivity and topological (weak) mixing of weighted composition operators on locally convex spaces of scalar valued functions which are defined by local properties. In Section 2 we introduce the setting of general locally convex sheaves of functions which gives the appropriate general framework for our objective. This general approach enables us to give an almost characterization of these properties in Section 3 (Theorems 3.9 and 3.11). For many concrete function spaces these almost characterizations can be made into characterizations with only a minimal additional effort. This is shown in Section 4 where we not only recover and unify most of the above mentioned results by our general approach but where our abstract setting also permits improving these results in one direction or another.

Moreover, as one of the main applications of our approach we show in Section 5 that topological transitivity and topological weak mixing for weighted composition operators $C_{w,\psi}$ on spaces of ultradifferentiable functions of Beurling type as well as of Roumieu type are equivalent, and we characterize these properties together with topological mixing in terms of the weight w and the symbol ψ (Theorems 5.2 and 5.3).

As another main application, we show in Section 6 that for weighted composition operators on kernels of elliptic partial differential operators in $C^\infty(X)$ for open subsets X of $\mathbb{R}^d$ which are homeomorphic to $\mathbb{R}^d$, hypercyclicity and topological weak mixing are equivalent whenever $|w| \leq 1$, and we again characterize these properties as well as topological mixing in terms of the weight function and the symbol of the operator (Theorem 6.2). We pay special attention to eigenspaces of the Laplace operator (Corollary 6.7) and of the Cauchy–Riemann operator (Corollary 6.8). In the latter case we show that weak mixing, hypercyclicity, and mixing for weighted composition operators coincide whenever $|w| \leq 1$. Additionally, we characterize in terms of the weight and symbol which weighted composition operators are well-defined on the eigenspaces of the Laplace operator and of the Cauchy–Riemann operator (Proposition 6.6).

Throughout, we use standard notation and terminology from functional analysis. For unexplained notions related to functional analysis we refer to [MeVo]. Moreover, we use common notation from the theory of linear partial differential operators. For this we refer to [Hö1]. Finally, for notions and results from the dynamics of linear operators which are not explained in the text we refer the reader to [BaMa2] and [GEPe].

2. Function spaces defined by local properties. In order to deal with weighted composition operators on several function spaces at once we choose the framework of sheaves. Here and throughout let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Definition 2.1. Let Ω be a locally compact, σ-compact, non-compact Hausdorff space and $\mathscr{F}$ a sheaf of functions on Ω, i.e.

- For each open subset $X \subseteq \Omega$, $\mathscr{F}(X)$ is a vector space of $\mathbb{K}$-valued functions, and if $Y \subseteq \Omega$ is another open set with $Y \subseteq X$, the restriction mapping
$$r_X^Y : \mathscr{F}(X) \to \mathscr{F}(Y), \qquad f \mapsto f_{|Y},$$
is well-defined.

- (Localization) For every open cover $(X_\iota)_{\iota\in I}$ of an open set $X \subseteq \Omega$ and for any $f, g \in \mathscr{F}(X)$ with $f_{|X_\iota} = g_{|X_\iota} (\iota \in I)$ we have $f = g$.
- (Gluing) For each open cover $(X_\iota)_{\iota\in I}$ of an open set $X \subseteq \Omega$ and for all $(f_\iota)_\iota \in \prod_{\iota\in I} \mathscr{F}(X_\iota)$ with $f_{\iota|X_\iota\cap X_\kappa} = f_{\kappa|X_\iota\cap X_\kappa}$ $(\iota, \kappa \in I)$ there is $f \in \mathscr{F}(X)$ with $f_{|X_\iota} = f_\iota\,(\iota \in I)$.

From these defining properties it follows immediately that for every open $X \subseteq \Omega$ and each open, relatively compact exhaustion $(X_n)_{n\in\mathbb{N}_0}$ of X the space $\mathscr{F}(X)$ and the projective limit $\operatorname{proj}_{\leftarrow n}(\mathscr{F}(X_n), r^{X_n}_{X_{n+1}})$, i.e. the subspace

$$\Big\{(f_n)_{n\in\mathbb{N}_0} \in \prod_{n\in\mathbb{N}_0} \mathscr{F}(X_n);\ \forall n \in \mathbb{N}_0 : f_n = r^{X_n}_{X_{n+1}}(f_{n+1})\Big\}$$

of $\prod_{n\in\mathbb{N}_0} \mathscr{F}(X_n)$, are algebraically isomorphic via the mapping

$$\mathscr{F}(X) \to \operatorname{proj}_{\leftarrow n}(\mathscr{F}(X_n), r^{X_n}_{X_{n+1}}), \qquad f \mapsto (r^{X_n}_X(f))_{n\in\mathbb{N}_0} = (f_{|X_n})_{n\in\mathbb{N}_0},$$

where injectivity follows from the localization property and surjectivity from the gluing property of a sheaf.

In order to be able to apply results from functional analysis, we define the following properties for a sheaf of functions $\mathscr{F}$ on Ω:

($\mathscr{F}$1) For every open subset $X \subseteq \Omega$ the function space $\mathscr{F}(X)$ is a webbed and ultrabornological Hausdorff locally convex space (which is satisfied, for example, if $\mathscr{F}(X)$ is a Fréchet space). Additionally, $\mathscr{F}(X) \subseteq C(X)$ for every open $X \subseteq \Omega$, and for each $x \in X$ the point evaluation δ_x at x is a continuous linear functional on $\mathscr{F}(X)$.
It then follows that for open $X, Y \subseteq \Omega$ with $Y \subseteq X$ the restriction map r^Y_X has closed graph, hence is continuous by De Wilde's Closed Graph Theorem (see e.g. [MeVo, Theorem 24.31]).
Moreover, we assume that for every open, relatively compact exhaustion $(X_n)_{n\in\mathbb{N}_0}$ of X the above algebraic isomorphism between $\mathscr{F}(X)$ and $\operatorname{proj}_{\leftarrow n}(\mathscr{F}(X_n), r^{X_n}_{X_{n+1}})$ is a homeomorphism.

($\mathscr{F}$2) For every compact $K \subseteq \Omega$ there is $f_K \in \mathscr{F}(\Omega)$ such that $f_K(x) \neq 0$ for each $x \in K$.

($\mathscr{F}$3) For any distinct $x, y \in \Omega$ there is $f \in \mathscr{F}(\Omega)$ with $f(x) = 0$ and $f(y) = 1$.

Remark 2.2. (i) For a sheaf $\mathscr{F}$ on Ω such that $\mathscr{F}(X)$ is a locally convex space and r^Y_X is continuous for any open $Y \subseteq X \subseteq \Omega$ (we then say that $\mathscr{F}$ is a *locally convex sheaf* on Ω) it follows immediately that for every open, relatively compact exhaustion $(X_n)_{n\in\mathbb{N}}$ of X the canonical isomorphism between $\mathscr{F}(X)$ and $\operatorname{proj}_{\leftarrow n}(\mathscr{F}(X_n), r^{X_n}_{X_{n+1}})$ is continuous. Therefore, if $\mathscr{F}$ is a locally convex sheaf of continuous functions such that $\mathscr{F}(X)$ is a Fréchet space for each open $X \subseteq \Omega$ on which δ_x is continuous for every $x \in X$, it follows from the Open Mapping Theorem and the fact that Fréchet spaces are ultrabornological (see e.g. [MeVo, Remark 24.15c)]) and webbed (see e.g. [MeVo, Corollary 24.29]) that ($\mathscr{F}1$) is satisfied.

(ii) For a sheaf $\mathscr{F}$ on Ω satisfying ($\mathscr{F}1$) it follows from $\delta_x \in \mathscr{F}(X)'$ for each $x \in X$ that the inclusion mapping

$$\mathscr{F}(X) \hookrightarrow C(X), \quad f \mapsto f,$$

has closed graph, where we equip $C(X)$ as usual with the compact-open topology. Since $\mathscr{F}(X)$ is supposed to be ultrabornological, it follows from De Wilde's Closed Graph Theorem that this inclusion is continuous, i.e. the topology carried by $\mathscr{F}(X)$ is finer than the compact-open topology.

(iii) If $\mathscr{F}$ contains the constant functions, property ($\mathscr{F}3$) means precisely that $\mathscr{F}(\Omega)$ separates points.

Example 2.3. (i) For a σ-compact, locally compact, non-compact Hausdorff space Ω the sheaf C of continuous functions satisfies ($\mathscr{F}1$)–($\mathscr{F}3$), i.e. for an open subset $X \subseteq \Omega$ let $C(X)$ be the space of $\mathbb{K}$-valued continuous functions on X equipped with the compact-open topology, that is, the locally convex topology defined by the family $\{\|\cdot\|_K;\ K \subset X \text{ compact}\}$ of seminorms, where for a compact subset $K \subset X$,

$$\forall f \in C(X): \quad \|f\|_K := \sup_{x \in K} |f(x)|.$$

Recall that locally compact spaces are completely regular (see e.g. [En, Theorem 3.3.1]), so that ($\mathscr{F}3$) is indeed satisfied.

(ii) For $\Omega = \mathbb{R}^d$ and $r \in \mathbb{N}_0 \cup \{\infty\}$ we denote by C^r the sheaf of r-times continuously differentiable functions, i.e. for every open $X \subseteq \mathbb{R}^d$ let $C^r(X)$ be the space of $\mathbb{K}$-valued functions which are r-times continuously differentiable. We equip $C^r(X)$ with the topology of local uniform convergence of all partial derivatives up to order r, i.e. the locally convex topology defined by the family $\{\|\cdot\|_{l,K};\ l \in \mathbb{N}_0,\ l < r+1,\ K \subset X \text{ compact}\}$ of seminorms, where for $l < r+1$ and $K \subset X$ compact,

$$\forall f \in C^r(X): \quad \|f\|_{l,K} := \sup_{|\alpha| \leq l} \sup_{x \in K} |\partial^\alpha f(x)|.$$

This makes $C^r(X)$ a separable Fréchet space and the sheaf C^r on $\mathbb{R}^d$ is easily seen to satisfy ($\mathscr{F}1$)–($\mathscr{F}3$).

(iii) For $\Omega = \mathbb{C}$ let $\mathscr{H}$ be the sheaf of holomorphic functions, i.e. for $X \subseteq \mathbb{C}$ let $\mathscr{H}(X)$ denote the space of holomorphic functions on X endowed with the compact-open topology. Then $\mathscr{H}(X)$ is a separable Fréchet space and it follows easily that ($\mathscr{F}1$)–($\mathscr{F}3$) are satisfied. A more general example is (v) below.

(iv) Let again $\Omega = \mathbb{R}^d$ and denote by $\mathscr{A}$ the sheaf of real analytic functions, that is, for open $X \subseteq \mathbb{R}^d$, $\mathscr{A}(X)$ is the space of real analytic functions of X. One apparent way of equipping $\mathscr{A}(X)$ with a locally convex topology is by considering the finest locally convex topology such that all the restriction maps

$$\mathscr{H}(U) \to \mathscr{A}(X), \quad f \mapsto f_{|X},$$

are continuous, where $U \subseteq \mathbb{C}^d$ is an arbitrary open set for which $X \subseteq U \cap \mathbb{R}^d$ and where $\mathscr{H}(U)$ is equipped with the compact-open topology. It is not hard to see that this (inductive) topology is Hausdorff. Another way of endowing $\mathscr{A}(X)$ with a natural locally convex topology is by taking the initial topology with respect to all restriction maps

$$r_K : \mathscr{A}(X) \to \mathscr{H}(K), \quad f \mapsto f_{|K},$$

where $K \subseteq X$ is an arbitrary compact set and $\mathscr{H}(K)$ denotes the space of germs of holomorphic functions on K equipped with the locally convex inductive limit topology $\mathscr{H}(K) = \operatorname{ind}_{m\in\mathbb{N}} \mathscr{H}^\infty(U_m)$, where $(U_m)_{m\in\mathbb{N}}$ is a (decreasing) basis of $\mathbb{C}^d$-neighborhoods of K and $\mathscr{H}^\infty(U_m)$ denotes the Banach space of bounded holomorphic functions on U_m equipped with the supremum norm. Choosing a compact exhaustion $(K_n)_{n\in\mathbb{N}}$ of X it follows that topologized in this way, $\mathscr{A}(X)$ equals the (topological) projective limit of the projective sequence $(\mathscr{H}(K_n))_{n\in\mathbb{N}}$ with restrictions as linking maps.

Again, it is not hard to see that the first (inductive) topology on $\mathscr{A}(X)$ is coarser than the second (projective) topology. That these two topologies actually coincide is a fundamental result due to Martineau [Mar]. Since $\mathscr{H}(U)$ is a Fréchet space, thus ultrabornological, it follows that $\mathscr{A}(X)$ is ultrabornological as the inductive limit of ultrabornological spaces. As an LB-space, $\mathscr{H}(K)$ is webbed for every compact $K \subseteq X$ and therefore $\mathscr{A}(X)$ is webbed as the countable projective limit of webbed spaces.

Now let $X \subseteq \mathbb{R}^d$ be open and let $(X_n)_{n\in\mathbb{N}_0}$ be an open, relatively compact exhaustion of X. Consider the continuous bijection

$$i : \mathscr{A}(X) \to \operatorname{proj}_{\leftarrow n}(\mathscr{A}(X_n), r^{X_n}_{X_{n+1}}), \quad f \mapsto (r^{X_n}_X(f))_{n\in\mathbb{N}_0}.$$

We show that i is open. By the projective description of the topology on $\mathscr{A}(X)$, for any zero neighborhood V in $\mathscr{A}(X)$ there is a compact subset K of X and an absolutely convex zero neighborhood W in $\mathscr{H}(K)$ such that $r_K^{-1}(W) \subseteq V$. Thus, for every complex neighborhood U of K there is $\delta > 0$ with

$$r_K^{-1}(\{g \in \mathscr{H}(K);\, g \in \mathscr{H}^\infty(U),\, \|g\|_{\infty,U} < \delta\}) \subseteq V,$$

where $\|\cdot\|_{\infty,U}$ denotes the supremum norm over U. If $m \in \mathbb{N}_0$ is chosen such that $K \subseteq X_m$, it follows with

$$\rho_K : \mathscr{A}(X_m) \to \mathscr{H}(K), \quad f \mapsto f_{|K},$$

that

$$V_m := \rho_K^{-1}(\{g \in \mathscr{H}(K);\, g \in \mathscr{H}^\infty(U),\, \|g\|_{\infty,U} < \delta\})$$

is a zero neighborhood in $\mathscr{A}(X_m)$. Clearly,

$$V_m \cap r_X^{X_m}(\mathscr{A}(X)) = r_X^{X_m}\big(r_K^{-1}(\{g \in \mathscr{H}(K);\, g \in \mathscr{H}^\infty(U),\, \|g\|_{\infty,U} < \delta\})\big) \\ \subseteq r_X^{X_m}(V).$$

If we set, for $k \in \mathbb{N}_0$,

$$\pi_k : \operatorname{proj}_{\leftarrow n}(\mathscr{A}(X_n), r_{X_{n+1}}^{X_n}) \to \mathscr{A}(X_k), \qquad (f_n)_{n\in\mathbb{N}_0} \mapsto f_k,$$

it follows that

$$i(V) = \pi_m^{-1}(r_X^{X_m}(V)) \supseteq \pi_m^{-1}\big(V_m \cap r_X^{X_m}(\mathscr{A}(X))\big) = \pi_m^{-1}(V_m),$$

so that $i(V)$ is a zero neighborhood in $\operatorname{proj}_{\leftarrow n}(\mathscr{A}(X_n), r_{X_{n+1}}^{X_n})$, which proves that i is open. Thus, the sheaf $\mathscr{A}$ on $\mathbb{R}^d$ satisfies $(\mathscr{F}1)$. Moreover, $(\mathscr{F}2)$ as well as $(\mathscr{F}3)$ are obviously satisfied, too.

(v) For $\Omega = \mathbb{R}^d$ and a complex polynomial P in d variables, i.e. $P \in \mathbb{C}[X_1, \dots, X_d]$, we define, for an open $X \subseteq \mathbb{R}^d$,

$$C_P^\infty(X) := \{f \in C^\infty(X);\, P(\partial)f = 0 \text{ in } X\},$$

where as usual for $P(\xi) = \sum_{|\alpha|\le m} a_\alpha \xi^\alpha$ we set

$$\forall f \in C^\infty(X),\, x \in X: \quad P(\partial)f(x) = \sum_{|\alpha|\le m} a_\alpha \partial^\alpha f(x).$$

Obviously, $C_P^\infty(X)$ is a subspace of $C^\infty(X)$, and since $P(\partial)$ is a continuous linear operator on the separable Fréchet space $C^\infty(X)$ it follows that $C_P^\infty(X)$ is a closed subspace of $C^\infty(X)$, thus a separable Fréchet space with the relative topology. It is easily seen that C_P^∞ is a sheaf on $\mathbb{R}^d$ which satisfies $(\mathscr{F}1)$. In order to see that $(\mathscr{F}2)$ holds as well, let $\zeta \in \mathbb{C}^d$ satisfy $P(\zeta) = 0$ (we exclude the rather boring case of a constant polynomial P). Then

$$e_\zeta : \mathbb{R}^d \to \mathbb{C}, \qquad e_\zeta(x) = \exp\Big(\sum_{j=1}^d \zeta_j x_j\Big),$$

belongs to $C_P^\infty(\mathbb{R}^d)$, which shows that $(\mathscr{F}2)$ is indeed satisfied.

However, $(\mathscr{F}3)$ need not hold for general (non-constant) P as is seen by taking $d = 2$ and $P(\xi_1, \xi_2) = \xi_1$. Then $C_P^\infty(X)$ consists obviously of functions which are independent of x_1 and there is no $f \in C_P^\infty(\mathbb{R}^2)$ with $f(0,0) = 1$ and $f(1,0) = 0$. However, by Proposition 6.1, $(\mathscr{F}3)$ is satisfied by C_P^∞ whenever P is (hypo)elliptic.

Considering the special cases of $P(\partial)$ being the Cauchy–Riemann operator, resp. the Laplace operator, gives as C_P^∞ the sheaf of holomorphic functions $\mathscr{H}$ on open subsets of $\mathbb{C}$, resp. the sheaf of harmonic functions on open subsets of $\mathbb{R}^d$.

In most of the above examples the sheaves considered are sheaves of C^1-functions over open subsets of euclidean space. For this kind of sheaves we introduce yet another property.

DEFINITION 2.4. Let $\Omega = \mathbb{R}^d$ and let $\mathscr{F}$ be a sheaf satisfying ($\mathscr{F}$1). Then we define:

($\mathscr{F}$4) For every open $X \subseteq \mathbb{R}^d$ we have $\mathscr{F}(X) \subseteq C^1(X)$ and for every $x \in X$, $1 \leq j \leq d$ the distribution of order one

$$-\partial_j \delta_x : \mathscr{F}(X) \to \mathbb{K}, \quad f \mapsto \partial_j f(x),$$

is continuous. Moreover, for each $h \in \mathbb{R}^d \setminus \{0\}, \lambda \in \mathbb{K}$, and $x \in X$ the kernel of the continuous linear functional

$$\mathscr{F}(X) \to \mathbb{K}, \quad f \mapsto \sum_{j=1}^{d} h_j \partial_j f(x) - \lambda f(x),$$

is not all of $\mathscr{F}(X)$.

Property ($\mathscr{F}$4) implies that for every direction $h \in \mathbb{R}^d \setminus \{0\}$ and any $x \in X$ the directional derivative in direction h evaluated at x does not coincide with a multiple of δ_x. Clearly, ($\mathscr{F}$4) is satisfied by Examples 2.3(ii) for $r \geq 1$, (iii), and (iv). For hypoelliptic polynomials P the sheaf C_P^∞ satisfies ($\mathscr{F}$4) by Proposition 6.1.

General assumption. Let $\mathscr{F}$ be a sheaf on Ω satisfying ($\mathscr{F}$1), $X \subseteq \Omega$ open, and let $w : X \to \mathbb{K}$ and $\psi : X \to X$ be continuous; w will be called a *weight* and ψ a *symbol*. We assume that the weighted composition operator

$$C_{w,\psi} : \mathscr{F}(X) \to \mathscr{F}(X), \quad f \mapsto w \cdot (f \circ \psi),$$

is well-defined. For every $x \in X$ we have $\delta_x \in \mathscr{F}(X)'$ by hypothesis, and it follows easily from the Hahn–Banach Theorem that the linear span of $\{\delta_x; x \in X\}$ is weak*-dense in $\mathscr{F}(X)'$. Since $\mathscr{F}(X)$ is Hausdorff, it follows that $C_{w,\psi}$ has closed graph. By ($\mathscr{F}$1) we conclude from De Wilde's Closed Graph Theorem [MeVo, Theorem 24.31] that $C_{w,\psi}$ is continuous.

3. Dynamical properties of weighted composition operators. In this section we give necessary and sufficient conditions for a weighted composition operator defined on a local space of functions to be topologically transitive or topologically (weakly) mixing. We will see that in many concrete cases these necessary and sufficient conditions coincide, thus they yield a characterization of the said properties.

DEFINITION 3.1. Let E be a locally convex space and T a continuous linear operator on E.

(i) T is called (*topologically*) *transitive* if for any pair of non-empty open subsets U, V of E there is $m \in \mathbb{N}$ such that $T^m(U) \cap V \neq \emptyset$.
(ii) T is called (*topologically*) *weakly mixing* if $T \oplus T$ is transitive on $E \oplus E$, i.e. if for every choice of non-empty open subsets U_j, V_j of E, $j = 1, 2$, there is $m \in \mathbb{N}$ such that $T^m(U_j) \cap V_j \neq \emptyset$.
(iii) T is called (*topologically*) *mixing* if for every pair U, V of non-empty open subsets of E there is $M \in \mathbb{N}$ such that $T^m(U) \cap V \neq \emptyset$ for every $m \geq M$.
(iv) T is called *hypercyclic* if there is $x \in E$ whose orbit under T, i.e. $\mathrm{orb}(T, x) := \{T^m x;\, m \in \mathbb{N}_0\}$, is dense in E.

REMARK 3.2. (i) Clearly, E has to be separable in order to support a hypercyclic operator. By Birkhoff's Transitivity Criterion [GEPe, Theorem 2.19] a continuous linear operator on a separable Fréchet space is transitive if and only if it is hypercyclic.

(ii) Obviously, every mixing operator is weakly mixing and every weakly mixing operator is transitive. In general, the converses are not true. While it is not too complicated to give an example of a weakly mixing operator which is not mixing (see e.g. [GEPe, Remark 4.10]), it is highly intricate to construct a hypercyclic operator (on a Banach space) which fails to be weakly mixing. Such an operator was constructed by De la Rosa and Read [DR] (see also [BaMa]) who thereby solved a problem posed by Herrero [He] which had remained open for more than fifteen years.

(iii) It follows immediately from the definition that every transitive operator has dense image.

We first give necessary conditions for a weighted composition operator defined on a local space of functions to be transitive. This result is inspired by [Pr, Lemma 3.1]. Before we state the condition we recall a definition for symbols.

DEFINITION 3.3. Let $X \subseteq \Omega$ be open and $\psi : X \to X$ be continuous.

(i) ψ is called *run-away* if for each compact subset K of X there is $m \in \mathbb{N}$ such that $\psi^m(K) \cap K = \emptyset$, where ψ^m denotes the m-fold iterate of ψ.
(ii) ψ is called *strong run-away* if for each compact subset K of X there is $M \in \mathbb{N}$ such that $\psi^m(K) \cap K = \emptyset$ whenever $m \geq M$.

Clearly, strong run-away implies run-away. The converse is true for $\psi : \mathbb{R} \to \mathbb{R}$ continuous and injective, as shown in [Pr, Lemma 4.1]. Moreover, for $\psi : X \to X$ holomorphic on a simply connected domain $X \subsetneq \mathbb{C}$, run-away and strong run-away are the same thing, due to the Riemann Mapping Theorem combined with the Denjoy–Wolff Iteration Theorem (see e.g. [Sh, Chapter 5]), and in fact these properties are equivalent to ψ having no fixed point. To the author's best knowledge, there is no example of a continuous

$\psi : X \to X$ on an open subset X of a locally compact, σ-compact, non-compact Hausdorff space Ω which is run-away but not strong run-away.

Proposition 3.4. *Let $\mathscr{F}$ be a sheaf on Ω satisfying $(\mathscr{F}1)$–$(\mathscr{F}3)$. Let $X \subseteq \Omega$ be non-empty and open and assume that the weighted composition operator $C_{w,\psi}$ is transitive on $\mathscr{F}(X)$. Then:*

(i) $\forall x \in X : w(x) \neq 0$.
(ii) *ψ is injective.*
(iii) *ψ has no fixed points.*
(iv) *$\forall x \in X : \overline{\{\psi^m(x);\, m \in \mathbb{N}_0\}}$ is not a compact subset of X, where the closure is taken in X.*
(v) *ψ is run-away under any of the following additional assumptions:*
    (v-1) $\forall x \in X : |w(x)| \leq 1$.
    (v-2) *$C_{w,\psi}$ is weakly mixing.*
    (v-3) *$\mathscr{F}(\Omega)$ is dense in $C(\Omega)$ with the compact-open topology.*

Additionally, if $\mathscr{F}$ satisfies $(\mathscr{F}4)$ then

(vi) *$\forall x \in X : \det J\psi(x) \neq 0$, where $J\psi(x)$ denotes the Jacobian of ψ at x.*

Proof. In order to prove (i), assume that $w(x_0) = 0$ for some $x_0 \in X$. Then the image of $C_{w,\psi}$ is contained in the kernel of δ_{x_0}, which is a closed subspace of $\mathscr{F}(X)$ due to $(\mathscr{F}1)$. By $(\mathscr{F}3)$ this closed subspace is proper, in particular nowhere dense. Hence, the image of $C_{w,\psi}$ is not dense in $\mathscr{F}(X)$, contradicting $C_{w,\psi}$ being transitive, so that (i) follows.

Next, we assume that $\psi(x) = \psi(y)$ for some distinct $x, y \in X$. Because $w(y) \neq 0$ by (i), it follows that

$$\operatorname{im} C_{w,\psi} \subseteq \operatorname{kern}\left(\delta_x - \frac{w(x)}{w(y)}\delta_y\right).$$

Since $w(x) \neq 0$ by (i), it follows from $(\mathscr{F}1)$ and $(\mathscr{F}3)$ that $\operatorname{kern}\big(\delta_x - \frac{w(x)}{w(y)}\delta_y\big)$ is a closed, proper subspace of $\mathscr{F}(X)$. Since the image of $C_{w,\psi}$ is dense in $\mathscr{F}(X)$, we obtain a contradiction as in the proof of (i), so that (ii) follows.

To prove (iii), for $\alpha \in \mathbb{K}$ and $r > 0$ we denote by $B(\alpha, r)$ the open ball about α with radius r and the corresponding closed ball by $B[\alpha, r]$. Assume there is $x_0 \in X$ with $\psi(x_0) = x_0$. If $|w(x_0)| \leq 1$, we have

$$\forall f \in \delta_{x_0}^{-1}(B(0,1)), n \in \mathbb{N}_0 : \quad |C_{w,\psi}^n(f)(x_0)| \leq 1,$$

and thus

$$\forall n \in \mathbb{N}_0 : \quad C_{w,\psi}^n(\delta_{x_0}^{-1}(B(0,1))) \cap \delta_{x_0}^{-1}(B(2,1)) = \emptyset. \tag{1}$$

But since $\operatorname{kern} \delta_{x_0} \neq \mathscr{F}(X)$ by $(\mathscr{F}3)$, we have $\operatorname{im} \delta_{x_0} = \mathbb{K}$, so $\delta_{x_0}^{-1}(B(2,1))$ is a non-empty, open (by $(\mathscr{F}1)$) subset of $\mathscr{F}(X)$, as is $\delta_{x_0}^{-1}(B(0,1))$. Hence,

(1) contradicts the transitivity of $C_{w,\psi}$. When $|w(x_0)| > 1$ we have

$$\forall f \in \delta_{x_0}^{-1}(\mathbb{K} \setminus B[0,1]),\, n \in \mathbb{N} : \quad |C_{w,\psi}^n(f)(x_0)| > 1,$$

implying

$$\forall n \in \mathbb{N}_0 : \quad C_{w,\psi}^n(\delta_{x_0}^{-1}(\mathbb{K} \setminus B[0,1])) \cap \delta_{x_0}^{-1}(B(0,1)) = \emptyset,$$

which again contradicts the transitivity of $C_{w,\psi}$. Thus, (iii) is proved.

In order to prove (iv), assume that for some $x_0 \in X$ the set

$$K := \overline{\{\psi^m(x_0);\, m \in \mathbb{N}_0\}}$$

is compact. By (i), there are $a, b > 0$ such that

$$\forall x \in K : \quad a \leq |w(x)| \leq b.$$

Fix f_K according to ($\mathscr{F}2$) and set

$$\alpha := \inf_{x \in K} |f_K(x)| \quad \text{and} \quad \beta := \sup_{x \in K} |f_K(x)|.$$

Then $\alpha > 0$ and $\beta < \infty$, and the set

$$U := \{f \in \mathscr{F}(X);\, \forall x \in K : \alpha/2 < |f(x)| < 2\beta\}$$

obviously contains f_K and is open in the compact-open topology. Thus, U is an open neighborhood of f_K in $\mathscr{F}(X)$. A straightforward calculation gives

$$\forall f \in U,\, m \in \mathbb{N}_0 : \quad \left| \frac{C_{w,\psi}^m(f)(\psi(x_0))}{C_{w,\psi}^m(f)(x_0)} \right| \leq \frac{4b\beta}{a\alpha}. \tag{2}$$

By ($\mathscr{F}1$) it follows that

$$V := \left\{ f \in \mathscr{F}(X);\, |\delta_{\psi(x_0)}(f)| > \frac{4b\beta}{a\alpha} |\delta_{x_0}(f)| \right\}$$

is an open subset of $\mathscr{F}(X)$, and since $\psi(x_0) \neq x_0$ by (iii), it follows from ($\mathscr{F}3$) that $V \neq \emptyset$. From (2) we obtain

$$\forall m \in \mathbb{N}_0 : \quad C_{w,\psi}^m(U) \cap V = \emptyset,$$

which contradicts the transitivity of $C_{w,\psi}$. Thus, (iv) is proved.

We continue with the proof of (v) and argue again by contradiction. Assume there is a compact subset K of X with

$$\forall m \in \mathbb{N}\, \exists x_m \in K : \quad \psi^m(x_m) \in K.$$

Due to ($\mathscr{F}2$) there is $f_K \in \mathscr{F}(X)$ with $f_K(x) \neq 0$ for every $x \in K$. We set

$$\alpha := \inf_{y \in K} |f_K(y)| > 0, \qquad \beta := \sup_{y \in K} |f_K(y)| < \infty.$$

We first assume that additionally $|w(x)| \leq 1$ for every $x \in X$. We define

$$U_1 := \{g \in \mathscr{F}(X);\, \forall x \in K : |g(x)| < \alpha/2\},$$
$$V_1 := \{g \in \mathscr{F}(X);\, \forall x \in K : |f_K(x) - g(x)| < \alpha/2\},$$

which are open in the compact-open topology and therefore, by ($\mathscr{F}1$), open subsets of $\mathscr{F}(X)$. Obviously, $f_K \in V_1$ and $0 \in U_1$. For every $m \in \mathbb{N}$ and each $g \in U_1$ it follows from $|w| \leq 1$ and $x_m, \psi^m(x_m) \in K$ that

$$\begin{aligned}\max_{x\in K} |f_K(x) - C_{w,\psi}^m(g)(x)| &\geq \Big| f_K(x_m) - \prod_{j=0}^{m-1} w(\psi^j(x_m)) g(\psi^m(x_m))\Big| \\ &\geq \alpha - |g(\psi^m(x_m))| > \alpha/2,\end{aligned}$$

so that $C_{w,\psi}^m(U_1) \cap V_1 = \emptyset$, which contradicts the transitivity of $C_{w,\psi}$.

Next we assume that $C_{w,\psi}$ is weakly mixing. We define

$$\begin{aligned}U_2 &:= \{g \in \mathscr{F}(X);\, \forall x \in K : \alpha/2 < |g(x)| < 2\beta\}, \\ V_2 &:= \Big\{g \in \mathscr{F}(X);\, \sup_{x\in K} |g(x)| < \alpha^2/(8\beta)\Big\},\end{aligned}$$

which are open in the compact-open topology and thus open subsets of $\mathscr{F}(X)$ by ($\mathscr{F}1$). Because $0 \in V_2$ and $f_K \in U_2$ and since $C_{w,\psi}$ is weakly mixing, there is $m \in \mathbb{N}$ such that

$$C_{w,\psi}^m(U_2) \cap U_2 \neq \emptyset \quad \text{and} \quad C_{w,\psi}^m(U_2) \cap V_2 \neq \emptyset.$$

Pick $f \in U_2$ with $C_{w,\psi}^m(f) \in U_2$. By $\psi^m(x_m) \in K$ we have $|f(\psi^m(x_m))| < 2\beta$, so that because of $x_m \in K$,

$$\begin{aligned}\alpha/2 < |C_{w,\psi}^m(f)(x_m)| &= \Big|\prod_{j=0}^{m-1} w(\psi^j(x_m)) f(\psi^m(x_m))\Big| \\ &\leq \Big|\prod_{j=0}^{m-1} w(\psi^j(x_m))\Big| 2\beta.\end{aligned} \tag{3}$$

On the other hand, for $g \in U_2$ such that $C_{w,\psi}^m(g) \in V_2$ it follows from $\psi^m(x_m) \in K$ and thus $|g(\psi^m(x_m))| > \alpha/2$ with $x_m \in K$ that

$$\frac{\alpha^2}{8\beta} > |C_{w,\psi}^m g(x_m)| = \Big|\prod_{j=0}^{m-1} w(\psi^j(x_m)) g(\psi^m(x_m))\Big| > \frac{\alpha}{2}\Big|\prod_{j=0}^{m-1} w(\psi^j(x_m))\Big|,$$

which contradicts (3).

In order to finish the proof of (v), we next assume that $\mathscr{F}(\Omega)$ is dense in $C(\Omega)$. Because of (i) there are $a, b > 0$ such that

$$\forall x \in K: \quad a \leq |w(x)| \leq b.$$

By ($\mathscr{F}2$) there is $\tilde{f}_K \in \mathscr{F}(X)$ such that $\tilde{f}_K(x) \neq 0$ for every $x \in K \cup \psi(K)$, so that

$$\tilde{\alpha} := \inf_{x\in K\cup\psi(K)} |\tilde{f}_K(x)| > 0 \quad \text{and} \quad \tilde{\beta} := \sup_{x\in K\cup\psi(K)} |\tilde{f}_K(x)| < \infty.$$

We define

$$U_3 := \{g \in \mathscr{F}(X);\, \forall x \in K \cup \psi(K) : \tilde{\alpha}/2 < |g(x)| < 2\tilde{\beta}\},$$

which is an open neighborhood of $\tilde{f}_K$ in $\mathscr{F}(X)$ satisfying

$$\forall g \in U_3, m \in \mathbb{N}_0 : \quad \left|\frac{C^m_{w,\psi}(g)(\psi(x_m))}{C^m_{w,\psi}(g)(x_m)}\right| \leq \frac{4b\tilde{\beta}}{a\tilde{\alpha}},$$

that is,

$$\forall g \in U_3, m \in \mathbb{N}_0 : \quad |\delta_{\psi(x_m)}(C^m_{w,\psi}(g))| \leq \frac{4b\tilde{\beta}}{a\tilde{\alpha}}|\delta_{x_m}(C^m_{w,\psi}(g))|. \tag{4}$$

In particular

$$\forall m \in \mathbb{N}_0 : \quad C^m_{w,\psi}(U_3) \cap (V_3 \cap \mathscr{F}(X)) = \emptyset,$$

where

$$V_3 := \left\{g \in C(X);\, \forall x \in K : |\delta_{\psi(x)}(g)| > \frac{4b\tilde{\beta}}{a\tilde{\alpha}}|\delta_x(g)|\right\}.$$

Once we show that $V_3 \cap \mathscr{F}(X)$ is a non-empty open subset of $\mathscr{F}(X)$, this will yield the desired contradiction to the transitivity of $C_{w,\psi}$.

A straightforward calculation shows that for $g, h \in C(X)$ and $x \in K$,

$$|\delta_{\psi(x)}(h)| - \frac{4b\tilde{\beta}}{a\tilde{\alpha}}|\delta_x(h)| \geq |\delta_{\psi(x)}(g)| - \frac{4b\tilde{\beta}}{a\tilde{\alpha}}|\delta_x(g)| - \frac{8b\tilde{\beta}}{a\tilde{\alpha}} \sup_{y \in K \cup \psi(K)} |g(y) - h(y)|,$$

which shows that V_3 is an open subset of $C(X)$ with the compact-open topology, thus $V_3 \cap \mathscr{F}(X)$ is open in $\mathscr{F}(X)$ by ($\mathscr{F}1$). Since $\mathscr{F}(\Omega)$ is dense in $C(\Omega)$ and $\{f_{|X};\, f \in C(\Omega)\}$ is dense in $C(X)$ by an application of Urysohn's Lemma, it is enough to show that V_3 is not empty in order to prove $V_3 \cap \mathscr{F}(X) \neq \emptyset$.

To show that $V_3 \neq \emptyset$ we use a clever construction from [Pr, Lemma 3.2]. Let $Y \subseteq X$ be an open, relatively compact neighborhood of K. By Urysohn's Lemma there is $h \in C(X)$ such that $h_{|K} = 0$, $h_{|X\setminus Y} = 1$ and $0 \leq h \leq 1$. The series

$$f := \sum_{m=0}^{\infty} \left(\frac{a\alpha}{4b\beta + a\alpha}\right)^m h \circ \psi^m$$

converges uniformly on X, in particular $f \in C(X)$ and because of (iv) it follows that $f(x) > 0$ for all $x \in X$. Moreover, for each $x \in K$ we have

$$|\delta_{\psi(x)}(f)| = f(\psi(x)) = \frac{4b\beta + a\alpha}{a\alpha} f(x) > \frac{4b\beta}{a\alpha}|\delta_x(f)|,$$

i.e. $f \in V_3$. Thus, (v) is proved.

Finally, let $\mathscr{F}$ satisfy ($\mathscr{F}4$) in addition to ($\mathscr{F}1$)–($\mathscr{F}3$). Assuming the existence of $x_0 \in X$ with $\det J\psi(x_0) = 0$ there is $h \in \mathbb{R}^d \setminus \{0\}$ in the kernel of $J\psi(x_0)$. By an easy calculation we have

$$\forall f \in \mathscr{F}(X): \quad \langle \nabla C_{w,\psi}(f)(x_0), h\rangle = C_{w,\psi}(f)(x_0)\frac{1}{w(x_0)}\langle \nabla w(x_0), h\rangle,$$

showing

$$\operatorname{im} C_{w,\psi} \subseteq \operatorname{kern}\Big(f \mapsto \langle \nabla f(x_0), h\rangle - \frac{1}{w(x_0)}\langle \nabla w(x_0), h\rangle \delta_{x_0}(f)\Big).$$

By ($\mathscr{F}4$) the previously mentioned kernel is a closed, proper subspace of $\mathscr{F}(X)$, which contradicts the image of $C_{w,\psi}$ being dense in $C_{w,\psi}$. ∎

DEFINITION 3.5. Let $\mathscr{F}$ be a sheaf on Ω satisfying ($\mathscr{F}1$) and $X \subseteq \Omega$ be open such that $C_{w,\psi}$ is well-defined on $\mathscr{F}(X)$. The operator $C_{w,\psi}$ is said to *act locally on* $\mathscr{F}(X)$ if for every open subset Y of X,

$$C_{w,\psi,Y} : \mathscr{F}(Y) \to \mathscr{F}(\psi^{-1}(Y)), \quad f \mapsto w \cdot (f \circ \psi),$$

i.e. $C_{w,\psi}$ (formally) applied to functions defined only on Y is well-defined.

REMARK 3.6. Clearly, under hypothesis ($\mathscr{F}1$), for every open subset Y of Ω and any $f \in \mathscr{F}(Y)$ the function

$$\psi^{-1}(Y) \to \mathbb{K}, \quad y \mapsto w(y)f(\psi(y)),$$

is a well-defined continuous function.

If $C_{w,\psi}$ operates locally on $\mathscr{F}(X)$ it follows from ($\mathscr{F}1$) and De Wildes's Closed Graph Theorem that $C_{w,\psi,Y}$ is continuous from $\mathscr{F}(Y)$ to $\mathscr{F}(\psi^{-1}(Y))$ for any open subset Y of X. It follows immediately from the definition of the restriction maps r_X^Y etc. and $C_{w,\psi,Y}$ that

$$C_{w,\psi,Y} \circ r_X^Y = r_X^{\psi^{-1}(Y)} \circ C_{w,\psi},$$

and more generally

$$\forall m \in \mathbb{N}: \quad C_{w,\psi,(\psi^{m-1})^{-1}(Y)} \circ \cdots \circ C_{w,\psi,Y} \circ r_X^Y = r_X^{(\psi^m)^{-1}(Y)} \circ C_{w,\psi}^m \tag{5}$$

for every open $Y \subseteq X$.

PROPOSITION 3.7. *Let $\mathscr{F}$ be a sheaf on Ω satisfying ($\mathscr{F}1$), and $X \subseteq \Omega$ open such that $C_{w,\psi}$ acts locally on $\mathscr{F}(X)$. Let $(X_n)_{n\in\mathbb{N}}$ be a relatively compact, open exhaustion of X. Assume that:*

(a) *For every open, relatively compact subset Y of X and any $m \in \mathbb{N}_0$,*

$$r_X^{(\psi^m)^{-1}(Y)}(\mathscr{F}(X)) \subseteq \overline{(C_{w,\psi,(\psi^{m-1})^{-1}(Y)} \circ \cdots \circ C_{w,\psi,Y})(\mathscr{F}(Y))},$$

where the closure is taken in $\mathscr{F}((\psi^m)^{-1}(Y))$.

(b) *There are $m, n \in \mathbb{N}$ with the following properties:*
   - (b1) *$\psi^m(X_n)$ is an open subset of X.*
   - (b2) *$X_n \cap \psi^m(X_n) = \emptyset$.*
   - (b3) *The restriction map $r_X^{X_n\cup\psi^m(X_n)}$ has dense range.*

Then for every $f, g \in \mathscr{F}(X)$ and any absolutely convex zero neighborhood U_n in $\mathscr{F}(X_n)$ we have

$$\emptyset \neq C^m_{w,\psi}(f + (r^{X_n}_X)^{-1}(U_n)) \cap (g + (r^{X_n}_X)^{-1}(U_n)).$$

Proof. Fix $f, g \in \mathscr{F}(X)$ and an absolutely convex zero neighborhood U_n in $\mathscr{F}(X_n)$. By (b1), $Y := \psi^m(X_n)$ is an open, relatively compact subset of X and $Z := (\psi^m)^{-1}(Y)$ is an open subset of X with $X_n \subseteq Z$, so that $(r^{X_n}_Z)^{-1}\big(\frac{1}{2}U_n\big)$ is a zero neighborhood in $\mathscr{F}(Z)$. By hypothesis (a), there is $\tilde{g} \in \mathscr{F}(Y)$ such that the (continuous) operator

$$T := C_{w,\psi,(\psi^{m-1})^{-1}(Y)} \circ \cdots \circ C_{w,\psi,Y}$$

satisfies

$$T(\tilde{g}) - r^Z_X(g) \in (r^{X_n}_Z)^{-1}\big(\tfrac{1}{2}U_n\big).$$

Because $X_n \cap \psi^m(X_n) = X_n \cap Y = \emptyset$ it follows from the gluing property of a sheaf that there is $f_n \in \mathscr{F}(X_n \cup Y)$ with

$$r^{X_n}_{X_n\cup Y}(f_n) = r^{X_n}_X(f) \quad \text{and} \quad r^Y_{X_n\cup Y}(f_n) = \tilde{g}.$$

By hypothesis (b3) there is $h \in \mathscr{F}(X)$ such that

$$r^{X_n\cup Y}_X(h) - f_n \in (r^{X_n}_{X_n\cup Y})^{-1}\big(\tfrac{1}{2}U_n\big)(r^{X_n}_Z \circ T \circ r^Y_{X_n\cup Y})^{-1}\big(\tfrac{1}{2}U_n\big).$$

Thus

$$\tfrac{1}{2}U_n \ni r^{X_n}_{X_n\cup Y}(r^{X_n\cup Y}_X(h) - f_n) = r^{X_n}_X(h) - r^{X_n}_{X_n\cup Y}(f_n) = r^{X_n}_X(h) - r^{X_n}_X(f),$$

so that

$$h \in f + (r^{X_n}_X)^{-1}\big(\tfrac{1}{2}U_n\big).$$

We will show that $C^m_{w,\psi}(h) \in g + (r^{X_n}_X)^{-1}(U_n)$. Indeed, using (5) we have

$$\begin{aligned}
&(r^{X_n}_X \circ C^m_{w,\psi})(h) - r^{X_n}_X(g)\\
&= (r^{X_n}_Z \circ r^Z_X \circ C^m_{w,\psi})(h) - (r^{X_n}_Z \circ T)(\tilde{g}) + (r^{X_n}_Z \circ T)(\tilde{g}) - r^{X_n}_Z \circ r^Z_X(g)\\
&= (r^{X_n}_Z \circ r^{(\psi^m)^{-1}(Y)}_X \circ C^m_{w,\psi})(h) - (r^{X_n}_Z \circ T \circ r^Y_{X_n\cup Y})(f_n) + r^{X_n}_Z(T(\tilde{g}) - r^Z_X(g))\\
&= (r^{X_n}_Z \circ C_{w,\psi,(\psi^{m-1})^{-1}(Y)} \circ \cdots \circ C_{w,\psi,Y} \circ r^Y_{X_n\cup Y} \circ r^{X_n\cup Y}_X)(h)\\
&\quad - (r^{X_n}_Z \circ T \circ r^Y_{X_n\cup Y})(f_n) + r^{X_n}_Z(T(\tilde{g}) - r^Z_X(g))\\
&= (r^{X_n}_Z \circ T \circ r^Y_{X_n\cup Y})(r^{X_n\cup Y}_X(h) - f_n) + r^{X_n}_Z(T(\tilde{g}) - r^Z_X(g))\\
&\in \tfrac{1}{2}U_n + \tfrac{1}{2}U_n \subseteq U_n,
\end{aligned}$$

so that $C^m_{w,\psi}(h) \in g + (r^{X_n}_X)^{-1}(U_n)$. Thus we have shown

$$C^m_{w,\psi}(f + (r^{X_n}_X)^{-1}(U_n)) \cap (g + (r^{X_n}_X)^{-1}(U_n)) \neq \emptyset. \quad \blacksquare$$

That condition (a) from the previous proposition is in particular satisfied if $C_{w,\psi}$ has dense range is the content of the next one.

Proposition 3.8. *Let $\mathscr{F}$ be a sheaf on Ω satisfying ($\mathscr{F}1$), and $X \subseteq \Omega$ open such that $C_{w,\psi}$ acts locally on $\mathscr{F}(X)$. Assume that $C_{w,\psi}$ has dense range. Then for every open subset Y of X and any $m \in \mathbb{N}_0$,*

$$r_X^{(\psi^m)^{-1}(Y)}(\mathscr{F}(X)) \subseteq \overline{(C_{w,\psi,(\psi^{m-1})^{-1}(Y)} \circ \cdots \circ C_{w,\psi,Y})(\mathscr{F}(Y))},$$

where the closure is taken in $\mathscr{F}((\psi^m)^{-1}(Y))$.

Proof. Fix an open subset Y of X and $m \in \mathbb{N}$. From the hypothesis on the range of $C_{w,\psi}$, the continuity of the restriction map $r_X^{(\psi^m)^{-1}(Y)}$, and the commutativity relation (5), it follows that

$$\begin{aligned} r_X^{(\psi^m)^{-1}(Y)}(\mathscr{F}(X)) &= r_X^{(\psi^m)^{-1}(Y)}(\overline{C_{w,\psi}^m(\mathscr{F}(X))}) \subseteq \overline{r_X^{(\psi^m)^{-1}(Y)}(C_{w,\psi}^m(\mathscr{F}(X)))} \\ &= \overline{(C_{w,\psi,(\psi^{m-1})^{-1}(Y)} \circ \cdots \circ C_{w,\psi,Y})(\mathscr{F}(Y))}. \ \blacksquare \end{aligned}$$

We now come to an almost characterization of weak mixing for weighted composition operators acting locally on $\mathscr{F}(X)$, where $\mathscr{F}$ is a sheaf of functions defined by local properties which satisfies ($\mathscr{F}1$)–($\mathscr{F}3$).

Theorem 3.9. *Let $\mathscr{F}$ be a sheaf on Ω satisfying ($\mathscr{F}1$)–($\mathscr{F}3$), let $X \subseteq \Omega$ be open, and assume that the weighted composition operator $C_{w,\psi}$ acts locally on $\mathscr{F}(X)$. Then, among the following,* (i)$\Rightarrow$(ii)$\Rightarrow$(iv). *Additionally, if $\mathscr{F}(\Omega)$ is dense in $C(\Omega)$ in the compact-open topology or if $|w(x)| \le 1$ for all $x \in X$, then* (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv).

(i) (a) *For any $m \in \mathbb{N}_0$ and every open, relatively compact $Y \subseteq \psi^m(X)$,*

$$r_X^{(\psi^m)^{-1}(Y)}(\mathscr{F}(X)) \subseteq \overline{(C_{w,\psi,(\psi^{m-1})^{-1}(Y)} \circ \cdots \circ C_{w,\psi,Y})(\mathscr{F}(Y))},$$

where the closure is taken in $\mathscr{F}((\psi^m)^{-1}(Y))$.

(b) *There is an open, relatively compact exhaustion $(X_n)_{n\in\mathbb{N}_0}$ of X such that for every $n \in \mathbb{N}_0$ there is $m \in \mathbb{N}$ such that:*

(b1) *$\psi^m(X_n)$ is open.*

(b2) *$X_n \cap \psi^m(X_n) = \emptyset$.*

(b3) *The restriction map $r_X^{X_n \cup \psi^m(X_n)}$ has dense range.*

(ii) *$C_{w,\psi}$ is weakly mixing on $\mathscr{F}(X)$.*

(iii) *$C_{w,\psi}$ is transitive on $\mathscr{F}(X)$.*

(iv) *Condition* (i)(a) *holds, w has no zeros, ψ is injective and run-away, and in case of ($\mathscr{F}4$) with w and ψ continuously differentiable, additionally* $\det J\psi(x) \neq 0$ *for every $x \in X$.*

Remark 3.10. (i) If (iv) is valid, then ψ is in particular run-away so that (b2) is satisfied for any open, relatively compact exhaustion $(X_n)_{n\in\mathbb{N}_0}$ of X for suitable m.

(ii) For $\Omega = \mathbb{R}^d$, if ψ is injective it follows from Brouwer's Invariance of Domain Theorem (see [Br, Corollary 19.9] for an even stronger result) that for any open subset Y of X and each $m \in \mathbb{N}_0$ the set $\psi^m(Y)$ is open. Thus, for $\Omega = \mathbb{R}^d$, the only obstruction to the equivalence of (i), (ii), and (iv) in Theorem 3.9 is the existence of a particular open, relatively compact exhaustion $(X_n)_{n\in\mathbb{N}}$ which satisfies (b3). In concrete situations, this obstruction is overcome by a suitable approximation result which—depending on the concrete sheaf of functions under consideration—can be trivial (e.g. in the case of continuous functions) or highly sophisticated (as in the case of holomorphic functions in several variables [Za]).

(iii) Although looking rather deterrent, condition (i)(a) in Theorem 3.9 is in most concrete situations fulfilled for zero-free w and injective as well as open ψ (the latter is redundant for $\Omega = \mathbb{R}^d$ by Brouwer's Invariance of Domain Theorem) because of the following. Under the stated hypothesis on w and ψ, for any open subset Y of $\psi^m(X)$, $m \in \mathbb{N}_0$, and each $f \in \mathscr{F}((\psi^m)^{-1}(Y))$ the function

$$\tilde{f}: Y \to \mathbb{K}, \qquad y \mapsto \left(\frac{f}{\prod_{j=0}^{m-1} w(\psi^j(\cdot))}\right) \circ (\psi^m)^{-1}(y),$$

is well-defined and continuous. If $\tilde{f} \in \mathscr{F}(Y)$, a straightforward calculation gives

$$(C_{w,\psi,(\psi^{m-1})^{-1}(Y)} \circ \cdots \circ C_{w,\psi,Y})(\tilde{f}) = f$$

in $(\psi^m)^{-1}(Y)$. But in many concrete examples, $\tilde{f} \in \mathscr{F}(Y)$ indeed holds—if also $\det J\psi(x) \neq 0$ for all $x \in X$. For $\mathscr{F} = C^r$ this follows from the fact that if $C_{w,\psi}$ is well-defined then $w \in C^r(X)$ and therefore $1/w \in C^r(X)$, ψ is C^r, too, and by the Inverse Function Theorem (see e.g. [Na, Theorem 1.3.2, Remark 1.3.11]) the same holds for $(\psi^m)^{-1}$ so that $\tilde{f} \in C^r(Y)$. The same arguments also hold for real analytic or holomorphic functions in several variables (see e.g. again [Na], resp. [FrGr, Theorem 7.5]). Thus, in many concrete examples, condition (i)(a) of Theorem 3.9 is redundant in part (iv).

Proof of Theorem 3.9. Assume (i) holds and let $V_j, W_j \subseteq \mathscr{F}(X)$ be nonempty and open, $j = 1,2$. We fix $f_j \in V_j$, $g_j \in W_j$, $j = 1,2$. Since by ($\mathscr{F}$1) we have $\mathscr{F}(X) = \mathrm{proj}_{\leftarrow n}(\mathscr{F}(X_n), r^{X_n}_{X_{n+1}})$ topologically, there is $n \in \mathbb{N}$ and an absolute convex zero neighborhood U_n in $\mathscr{F}(X_n)$ such that for $j = 1,2$,

$$f_j + (r^{X_n}_X)^{-1}(U_n) \subseteq V_j \quad \text{and} \quad g_j + (r^{X_n}_X)^{-1}(U_n) \subseteq W_j$$

(see e.g. [We, Chapter 3.3]). From hypotheses (a) and (b) it follows with the aid of Proposition 3.7 that there is $m \in \mathbb{N}$ such that for $j = 1,2$,

$$\emptyset \neq C^m_{w,\psi}(f_j + (r^{X_n}_X)^{-1}(U_n)) \cap (g_j + (r^{X_n}_X)^{-1}(U_n)) \subseteq C^m_{w,\psi}(V_j) \cap W_j,$$

so that (ii) holds.

(ii) obviously implies (iii).

If (ii) is satisfied (respectively if (iii) is satisfied and $\mathscr{F}(\Omega)$ is dense in $C(\Omega)$ or $|w| \leq 1$) then $C_{w,\psi}$ has in particular dense range, so that condition (i)(a) follows from Proposition 3.8, while the rest of the properties listed in (iv) follow from Proposition 3.4. ∎

Concrete applications of Theorem 3.9 are postponed to Sections 4, 5, and 6. We next come to an almost characterization of mixing of weighted composition operators on local function spaces.

THEOREM 3.11. *Let $\mathscr{F}$ be a sheaf on Ω satisfying ($\mathscr{F}$1)–($\mathscr{F}$3), let $X \subseteq \Omega$ be open, and assume that the weighted composition operator $C_{w,\psi}$ acts locally on $\mathscr{F}(X)$. Then, among the following,* (i)⇒(ii)⇒(iii)*:*

(i) (a) *For any $m \in \mathbb{N}_0$ and every open, relatively compact $Y \subseteq \psi^m(X)$,*

$$r_X^{(\psi^m)^{-1}(Y)}(\mathscr{F}(X)) \subseteq \overline{(C_{w,\psi,(\psi^{m-1})^{-1}(Y)} \circ \cdots \circ C_{w,\psi,Y})(\mathscr{F}(Y))},$$

where the closure is taken in $\mathscr{F}((\psi^m)^{-1}(Y))$.

(b) *There is an open, relatively compact exhaustion $(X_n)_{n\in\mathbb{N}_0}$ of X such that for every $n \in \mathbb{N}_0$ there is $M \in \mathbb{N}$ such that:*

(b1) *$\psi^m(X_n)$ is open for all $m \geq M$.*

(b2) *$X_n \cap \psi^m(X_n) = \emptyset$ for all $m \geq M$.*

(b3) *$r_X^{X_n \cup \psi^m(X_n)}$ has dense range for all $m \geq M$.*

(ii) *$C_{w,\psi}$ is mixing on $\mathscr{F}(X)$.*

(iii) *Condition* (i)(a) *holds, w has no zeros, ψ is injective and strong run-away, and in case of ($\mathscr{F}$4) with w and ψ continuously differentiable, additionally* $\det J\psi(x) \neq 0$ *for every $x \in X$.*

Proof. In order to show that (i) implies (ii) let $V, W \subseteq \mathscr{F}(X)$ be open and non-empty. As in the proof of (i)⇒(ii) of Theorem 3.9, let $f \in V$, $g \in W$, and choose $n \in \mathbb{N}$ and an absolutely convex zero neighborhood U_n in $\mathscr{F}(X_n)$ such that

$$f + (r_X^{X_n})^{-1}(U_n) \subseteq V \quad \text{and} \quad g + (r_X^{X_n})^{-1}(U_n) \subseteq W.$$

From the hypotheses (a) and (b1)–(b3) together with Proposition 3.7 it follows that there is $M \in \mathbb{N}$ such that for any $m \geq M$,

$$\emptyset \neq C_{w,\psi}^m(f + (r_X^{X_n})^{-1}(U_n)) \cap (g + (r_X^{X_n})^{-1}(U_n)) \subseteq C_{w,\psi}^m(V) \cap W,$$

so that $C_{w,\psi}$ is mixing.

Conversely, if $C_{w,\psi}$ is mixing, then it is in particular weakly mixing, so by Theorem 3.9 we only have to show that ψ is strong run-away. Assume otherwise, i.e. there is a compact subset K of X and a strictly increasing sequence $(m_l)_{l\in\mathbb{N}}$ of natural numbers such that

$$\forall l \in \mathbb{N}\ \exists x_l \in K: \quad \psi^{m_l}(x_l) \in K.$$

By ($\mathscr{F}2$) there is $f_K \in \mathscr{F}(X)$ such that $f_K(x) \neq 0$ for all x in K. Then

$$\alpha := \inf_{x\in K} |f_K(x)| > 0 \quad \text{and} \quad \beta := \sup_{x\in K} |f_K(x)| < \infty,$$

and the set

$$U := \{g \in \mathscr{F}(X);\, \forall x \in K : \alpha/2 < |g(x)| < 2\beta\}$$

contains f_K and is open in the compact-open topology and is therefore an open neighborhood of f_K in $\mathscr{F}(X)$. The set

$$V := \left\{g \in \mathscr{F}(X);\, \sup_{x\in K} |g(x)| < \frac{\alpha^2}{8\beta}\right\}$$

is open in $\mathscr{F}(X)$, too, and contains the zero function. Since $C_{w,\psi}$ is mixing, there is $M \in \mathbb{N}$ such that

$$\forall m \geq M : \quad C_{w,\psi}^m(U) \cap U \neq \emptyset \quad \text{and} \quad C_{w,\psi}^m(U) \cap V \neq \emptyset.$$

Now we fix $l \in \mathbb{N}$ with $m_l > M$ and pick $f \in U$ with $C_{w,\psi}^{m_l}(f) \in U$ as well as $g \in U$ with $C_{w,\psi}^{m_l}(g) \in V$. As in the proof of Proposition 3.4(v), under the additional assumption (v-2) one deduces the contradiction

$$\frac{\alpha}{4\beta} < \Big|\prod_{j=0}^{m_l-1} w(\psi^j(x_l))\Big| < \frac{\alpha}{4\beta}.$$

Thus, ψ is strong run-away. ∎

4. Dynamics of weighted composition operators on concrete local function spaces. As a first application of the results from the previous section we show how to use them to recover characterizations of transitivity/hypercyclicity and mixing of weighted composition operators on various concrete function spaces obtained by several authors or which we assume to be well-known (and add a slight generalization here and there).

4.1. Continuous functions. For an arbitrary locally compact, σ-compact, non-compact Hausdorff space Ω the sheaf C of continuous functions satisfies properties ($\mathscr{F}1$)–($\mathscr{F}3$), as explained in Example 2.3(i). Clearly, for an arbitrary open $X \subseteq \Omega$, $w \in C(X)$, and continuous $\psi : X \to X$ the operator $C_{w,\psi}$ is well-defined on $C(X)$ and acts locally on $C(X)$.

Recall that locally compact spaces are completely regular, and that by [Wa, Theorem 5] for a completely regular topological space Z the space $C(Z)$ equipped with the compact-open topology is separable if and only if Z has a separable metrizable compression, i.e. if and only if Z has a weaker separable metrizable topology. Thus, if the open subset X of Ω below has a weaker separable metrizable topology, part (a) of the next result characterizes hypercyclicity of $C_{w,\psi}$ on $C(X)$.

Corollary 4.1. *Let Ω be a locally compact, σ-compact, non-compact Hausdorff space, $X \subseteq \Omega$ be open, $w \in C(X)$ and $\psi : X \to X$ be continuous. If $\Omega \neq \mathbb{R}^d$, we assume additionally that ψ is open.*

(a) *The following are equivalent:*

  (i) *$C_{w,\psi}$ is weakly mixing on $C(X)$.*
  (ii) *$C_{w,\psi}$ is transitive on $C(X)$.*
  (iii) *w has no zeros, ψ is injective and run-away.*

(b) *The following are equivalent:*

  (i) *$C_{w,\psi}$ is mixing on $C(X)$.*
  (ii) *w has no zeros, ψ is injective and strong run-away.*

Proof. We first prove (a). Clearly, (i) implies (ii) and by Theorem 3.9, (ii) implies (iii). Now, if (iii) is satisfied it follows from Remark 3.10 that condition (i)(a) from Theorem 3.9 is fulfilled. Let $(X_n)_{n\in\mathbb{N}_0}$ be an arbitrary open, relatively compact exhaustion of X. If $\Omega = \mathbb{R}^d$ it follows from Brouwer's Invariance of Domain Theorem that ψ^m is an open mapping on X for any m, in particular, for every $n \in \mathbb{N}_0$ and any $m \in \mathbb{N}$, $\psi^m(X_n)$ is an open subset of X. For $\Omega \neq \mathbb{R}^d$ the same follows from the hypotheses on ψ. For fixed $n \in \mathbb{N}_0$, since ψ is run-away, there is $m \in \mathbb{N}$ such that

$$X_n \cap \psi^m(X_n) = \emptyset.$$

Let K be a compact subset of $X_n \cup \psi^m(X_n)$. Since X is a locally compact Hausdorff space, it is in particular regular. Thus, every $x \in K$ has an open neighborhood V_x such that $\overline{V_x} \subseteq X_n \cup \psi^m(X_n)$. By the compactness of K there is an open neighborhood V of K such that $\overline{V} \subseteq X_n \cup \psi^m(X_n)$. By Urysohn's Lemma there is $h \in C(X)$ such that $h = 1$ on K and $h = 0$ on $X \setminus V$, thus $\operatorname{supp} h \subseteq \overline{V}$. Now if $g \in C(X_n \cup \psi^m(X_n))$ we obtain a continuous function f on X with $f_{|K} = g$ by extending hg by zero outside of $X_n \cup \psi^m(X_n)$. Since $K \subset X_n \cup \psi^m(X_n)$ was an arbitary compact set it follows that $r_X^{X_n\cup\psi^m(X_n)}$ has dense range. Hence, condition (i)(b) from Theorem 3.9 is also satisfied, so that (iii) implies (i).

Referring to Theorem 3.11 instead of Theorem 3.9, the proof of (b) is mutatis mutandis the same as the proof of (a). ∎

4.2. C^r-functions on open subsets of $\mathbb{R}^d$. Let $\mathscr{F}$ be the sheaf C^r of r-times continuously differentiable functions on $\mathbb{R}^d$ (equipped with the topology of local uniform convergence of partial derivatives of order less than $r+1$). Then C^r satisfies ($\mathscr{F}$1)–($\mathscr{F}$4) as explained in Example 2.3(ii), and $C^r(X)$ is a separable Fréchet space for every open $X \subseteq \mathbb{R}^d$. Clearly, for an arbitrary open $X \subseteq \mathbb{R}^d$, $w \in C^r(X)$, and a C^r-function $\psi : X \to X$ the weighted composition operator $C_{w,\psi}$ is well-defined on $C^r(X)$ and acts locally on $C^r(X)$. The next application of the results from the previous section gives the results obtained by Przestacki [Pr] for $r = \infty$.

COROLLARY 4.2. *Let* $X \subseteq \mathbb{R}^d$ *be open,* $r \in \mathbb{N} \cup \{\infty\}$, $w \in C^r(X)$ *and* $\psi : X \to X$ *be a* C^r*-function.*

(a) *The following are equivalent:*
   (i) $C_{w,\psi}$ *is weakly mixing on* $C^r(X)$.
   (ii) $C_{w,\psi}$ *is hypercyclic on* $C^r(X)$.
   (iii) w *has no zeros,* ψ *is injective, run-away, and* $\det J\psi(x) \neq 0$ *for all* $x \in X$.

(b) *The following are equivalent:*
   (i) $C_{w,\psi}$ *is mixing on* $C^r(X)$.
   (ii) w *has no zeros,* ψ *is injective, strong run-away, and* $\det J\psi(x) \neq 0$ *for all* $x \in X$.

Proof. We first prove (a). Since $C^r(X)$ is a separable Fréchet space, by Birkhoff's transitivity criterion, hypercyclicity of $C_{w,\psi}$ is equivalent to transitivity. Thus, (i) implies (ii) and since the polynomials, a fortiori $C^r(\mathbb{R}^d)$, are dense in $C(\mathbb{R}^d)$ (see e.g. [Tr, Chapter 15, Corollary 4]) by Theorem 3.9, (ii) implies (iii). Now, if (iii) is satisfied it follows from Remark 3.10 that condition (i)(a) from Theorem 3.9 is fulfilled. Let $(X_n)_{n\in\mathbb{N}_0}$ be an arbitrary open, relatively compact exhaustion of X. Because $\det J(x) \neq 0$ for all $x \in X$ it follows together with the injectivity of ψ that ψ^m is an open mapping (see e.g. [Na, Theorem 1.3.2]). In particular, for all $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$, $\psi^m(X_n)$ is an open subset of X. For fixed $n \in \mathbb{N}_0$, since ψ is run-away, there is $m \in \mathbb{N}$ such that

$$X_n \cap \psi^m(X_n) = \emptyset.$$

Let K be a compact subset of $X_n \cup \psi^m(X_n)$ and V an open neighborhood of K such that $\overline{V} \subseteq X_n \cup \psi^m(X_n)$. Then there is $h \in C^\infty(X)$ such that $h = 1$ on K and $\operatorname{supp} h \subseteq \overline{V}$. Now if $g \in C^\infty(X_n \cup \psi^m(X_n))$ we obtain a smooth function f on X with $f_{|K} = g$ by extending hg by zero outside of $X_n \cup \psi^m(X_n)$. Since $K \subseteq X_n \cup \psi^m(X_n)$ was an arbitrary compact set we conclude that $r_X^{X_n\cup\psi^m(X_n)}$ has dense range, so that (i)(b) from Theorem 3.9 is fulfilled. Thus, (iii) implies (i).

Referring again to Theorem 3.11 instead of Theorem 3.9, the proof of (b) is mutatis mutandis the same as that of (a). ∎

4.3. Real analytic functions on open subsets of $\mathbb{R}^d$. Let $\mathscr{F}$ be the sheaf $\mathscr{A}$ of real analytic functions on $\mathbb{R}^d$ (equipped with its natural topology, see Example 2.3(v)). Then $\mathscr{A}$ satisfies ($\mathscr{F}$1)–($\mathscr{F}$4) as explained in Example 2.3(v), and clearly, for an arbitrary open $X \subseteq \mathbb{R}^d$, $w \in \mathscr{A}(X)$, and real analytic $\psi : X \to X$, the operator $C_{w,\psi}$ is well-defined on $\mathscr{A}(X)$ and acts locally on $\mathscr{A}(X)$. For the special case of $w = 1$, the equivalence of (ii) and (iii) in part (a) of our next result was obtained by Bonet and Domański [BoDo, Theorem 2.3].

COROLLARY 4.3. *Let $X \subseteq \mathbb{R}^d$ be open, $w \in \mathscr{A}(X)$ and $\psi : X \to X$ be real analytic.*

(a) *The following are equivalent:*

   (i) *$C_{w,\psi}$ is weakly mixing on $\mathscr{A}(X)$.*
   (ii) *$C_{w,\psi}$ is transitive on $\mathscr{A}(X)$.*
   (iii) *w has no zeros, ψ is injective, run-away, and* $\det J\psi(x) \neq 0$ *for all $x \in X$.*

(b) *The following are equivalent:*

   (i) *$C_{w,\psi}$ is mixing on $\mathscr{A}(X)$.*
   (ii) *w has no zeros, ψ is injective, strong run-away, and* $\det J\psi(x) \neq 0$ *for all $x \in X$.*

Proof. As before, we first prove (a). Obviously, (i) implies (ii) and because the polynomials are dense in $C(\mathbb{R}^d)$, by Theorem 3.9, (ii) implies (iii). Now, if (iii) is satisfied it follows from Remark 3.10 that condition (i)(a) from Theorem 3.9 is fulfilled. Let $(X_n)_{n\in\mathbb{N}_0}$ be an arbitrary open, relatively compact exhaustion of X. As in the proof of Corollary 4.2, for all $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$, $\psi^m(X_n)$ is an open subset of X. For fixed $n \in \mathbb{N}_0$, since ψ is run-away, there is $m \in \mathbb{N}$ such that

$$X_n \cap \psi^m(X_n) = \emptyset,$$

so that (i)(b1, b2) in Theorem 3.9 are fulfilled.

In order to show that (b3) is fulfilled, too, let $f \in \mathscr{A}(X_n \cup \psi^m(X_n))$ be arbitrary and let V be any neighborhood of f in $\mathscr{A}(X_n \cup \psi^m(X_n))$. By the definition of the topology on $\mathscr{A}(X_n \cup \psi^m(X_n))$ there is a compact subset K of $X_n \cup \psi^m(X_n)$ and a complex neighborhood $W_0 \subseteq \mathbb{C}^d$ of K such that f extends to a holomorphic function on W_0 and for every complex neighborhood W of K with $\overline{W} \subseteq W_0$ there is $\delta > 0$ with

$$\{g \in \mathscr{H}^\infty(W);\ \|f - g\|_{\infty,W} < \delta\} \subseteq V,$$

where $\|\cdot\|_{\infty,W}$ denotes the supremum norm over W. Because compact subsets of $\mathbb{R}^d$ are polynomially convex in $\mathbb{C}^d$ it follows from [Hö2, Theorem 2.7.7] that for any relatively compact, complex neighborhood W of K with $\overline{W} \subseteq W_0$ there is a (complex) polynomial p such that $\|f - p\|_{\infty,W} < \delta$. In particular, $p_{|X} \in V \cap \mathscr{A}(X)$. By the arbitrariness of V and f the restriction map $r_X^{X_n \cup \psi^m(X_n)}$ has dense range, so that (i)(b3) in Theorem 3.9 is indeed fulfilled. Now, by the same theorem, (iii) implies (i) so that (a) is proved.

The proof of part (b) is again similar by referring to Theorem 3.11 instead of Theorem 3.9. ∎

A generalization of the setting of real analytic functions is given in the next section.

5. Spaces of ultradifferentiable functions. In this section we consider spaces of ultradifferentiable functions on open subsets of $\mathbb{R}^d$, both of Roumieu type and Beurling type, and both quasianalytic and non-quasianalytic classes.

There are at least two ways of defining spaces of ultradifferentiable functions. The classical Denjoy–Carleman classes are defined as smooth functions satisfying certain growth conditions on their Taylor coefficients, while it was observed by Beurling [Beu] (see also Björck [Bj]) that one can also use decay properties with respect to a weight function of the Fourier transform of compactly supported smooth functions. The latter approach was vastly generalized by Braun, Meise, and Taylor [BrMeTa]. It is their approach that we follow here. For a comparison of the two approaches, see [BoMeMe]; see also the article [RaSc1] by Rainer and Schindl.

Recall that a continuous increasing function $\omega : [0,\infty) \to [0,\infty)$ satisfying $\omega_{|[0,1]} = 0$ is called a *weight function* if the following properties hold:

(α) There is $K \geq 1$ such that $\omega(2t) \leq K(1+\omega(t))$ for all $t \geq 0$.
(β) $\omega(t) = O(t)$ as $t \to \infty$.
(γ) $\log(1+t) = o(\omega(t))$ as $t \to \infty$.
(δ) $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(x) := \omega(e^x)$ is convex.

Recall that a weight function ω is called *quasianalytic* if

(q) $\int_1^\infty \frac{\omega(t)}{t^2}\,dt = \infty$.

A weight function which does not satisfy (q) is called *non-quasianalytic.* Because of (γ) and (δ), for a weight function ω and φ as in (δ), the *Young conjugate* φ^* of φ

$$\varphi^* : [0,\infty) \to [0,\infty), \qquad \varphi^*(y) := \sup_{x\geq 0}(yx - \varphi(x)),$$

is well-defined, convex, increasing, and satisfies $\varphi^*(0) = 0$, $\lim_{y\to\infty} \frac{y}{\varphi^*(y)} = 0$, and $(\varphi^*)^* = \varphi$. For a weight function ω and an open $X \subseteq \mathbb{R}^d$ we define

$$\mathscr{E}_{(\omega)}(X) := \Big\{ f \in C^\infty(X) : \forall K \subseteq X \text{ compact } \forall m \in \mathbb{N} :$$
$$\|f\|_{(\omega),K,m} := \sup_{x\in K} \sup_{\alpha\in\mathbb{N}_0^d} |\partial^\alpha f(x)| \exp\Big(-m\varphi^*\Big(\frac{|\alpha|}{m}\Big)\Big) < \infty \Big\}$$

and

$$\mathscr{E}_{\{\omega\}}(X) := \Big\{ f \in C^\infty(X) : \forall K \subseteq X \text{ compact } \exists m \in \mathbb{N} :$$
$$\|f\|_{\{\omega\},K,m} := \sup_{x\in K} \sup_{\alpha\in\mathbb{N}_0^d} |\partial^\alpha f(x)| \exp\Big(-\frac{1}{m}\varphi^*(m|\alpha|)\Big) < \infty \Big\}.$$

The elements of $\mathscr{E}_{(\omega)}(X)$, resp. $\mathscr{E}_{\{\omega\}}(X)$, are called *$\omega$-ultradifferentiable functions of Beurling type on X*, resp. *ω-ultradifferentiable functions of Roumieu type on X*. Obviously, $\mathscr{E}_{(\omega)}(X) \subseteq \mathscr{E}_{\{\omega\}}(X)$, and clearly $\mathscr{E}_{(\omega)}$ and $\mathscr{E}_{\{\omega\}}$ are sheaves on $\mathbb{R}^d$. $\mathscr{E}_{(\omega)}(X)$ contains non-trivial functions with compact support for some non-empty open X if and only if ω is non-quasianalytic.

Prominent examples of weights are $\omega_\beta(t) = t^\beta$ with $0 < \beta < 1$ for which $\varphi^*(y) = y/\beta \log(y/e\beta)$ so that $\exp(-\varphi^*(|\alpha|\lambda)/\lambda) = (\lambda/\beta)^{-|\alpha|/\beta}(|\alpha|/e)^{-|\alpha|/\beta}$. By Stirling's formula $\mathscr{E}_{\{\omega_\beta\}}(X)$ is the classical *Gevrey class* of exponent $1/\beta$, denoted by $\Gamma^{1/\beta}(X)$, and $\mathscr{E}_{(\omega_\beta)}(X)$ is the so-called *small Gevrey class* of exponent $1/\beta$, denoted by $\gamma^{1/\beta}(X)$. The spaces $\Gamma^{1/\beta}(X)$ play an important role in the regularity theory of solutions of hypoelliptic partial differential equations [Hö1, Section 11.4].

Moreover, for the weight function $\omega(t) = t$ the Roumieu space $\mathscr{E}_{\{\omega\}}(X)$ coincides with $\mathscr{A}(X)$, while the Beurling space $\mathscr{E}_{(\omega)}(X)$ consists of the restrictions to X of functions from $\mathscr{H}(\mathbb{C}^d)$.

As usual, $\mathscr{E}_{(\omega)}(X)$ will be equipped with the locally convex topology induced by the family $\{\|\cdot\|_{(\omega),K,m};\ K \subseteq X \text{ compact}, m \in \mathbb{N}\}$ of seminorms, and $\mathscr{E}_{\{\omega\}}(X)$ will be topologized as $\operatorname{proj}_{\leftarrow Y} \operatorname{ind}_{m\to\infty} \mathscr{E}_{\{\omega\}}(Y,m)$, where for each open, relatively compact subset Y of X,

$$\mathscr{E}_{\{\omega\}}(Y,m) := \Big\{ f \in C^\infty(Y);$$
$$\|f\|_{\{\omega\},Y,m} := \sup_{x\in Y} \sup_{\alpha\in\mathbb{N}_0^d} |\partial^\alpha f(x)| \exp\Big(-\frac{1}{m}\varphi^*(m|\alpha|)\Big) < \infty \Big\}$$

endowed with the norm $\|\cdot\|_{\{\omega\},Y,m}$ is a Banach space.

PROPOSITION 5.1. *Let ω be a weight function. Equipped with their usual locally convex topologies, the sheaves $\mathscr{E}_{(\omega)}$ and $\mathscr{E}_{\{\omega\}}$ on $\mathbb{R}^d$ both satisfy properties ($\mathscr{F}1$)–($\mathscr{F}4$).*

Proof. It is well-known that for open $X \subseteq \mathbb{R}^d$ the space $\mathscr{E}_{(\omega)}(X)$ is a (nuclear) Fréchet space [BrMeTa, Proposition 4.9]. Obviously, the point evaluation δ_x is a continuous linear functional on $\mathscr{E}_{(\omega)}(X)$ for any $x \in X$. Therefore, as observed in Remark 2.2(i), the sheaf $\mathscr{E}_{(\omega)}$ on $\mathbb{R}^d$ satisfies ($\mathscr{F}1$). Moreover, since $\mathscr{E}_{(\omega)}(X)$ is closed under differentiation and since all polynomials obviously belong to $\mathscr{E}_{(\omega)}(X)$, properties ($\mathscr{F}2$)–($\mathscr{F}4$) are fulfilled, too.

Clearly, in the definition of the topology of $\mathscr{E}_{\{\omega\}}(X)$ it is enough to take the projective limit with respect to an open, relatively compact exhaustion $(Y_n)_{n\in\mathbb{N}}$ of X. Therefore, being the projective limit of a sequence of LB-spaces, $\mathscr{E}_{\{\omega\}}(X)$ is webbed. It has been shown recently by Debrouwere and Vindas [DeVi, Proposition 3.2] that the space of ultradifferentiable functions of Roumieu type is ultrabornological.

Let $(X_n)_{n\in\mathbb{N}_0}$ be an open, relatively compact exhaustion of X. In order to show that the continuous bijection

$$i : \mathscr{E}_{\{\omega\}}(X) \to \operatorname{proj}_{\leftarrow n}(\mathscr{E}_{\{\omega\}}(X_n), r^{X_n}_{X_{n+1}}), \qquad f \mapsto (r^{X_n}_X(f))_{n\in\mathbb{N}_0},$$

is open, let V be an arbitrary zero neighborhood in $\mathscr{E}_{\{\omega\}}(X)$. By the definition of the topology on $\mathscr{E}_{\{\omega\}}(X)$ there are $n \in \mathbb{N}$ and a zero neighborhood U_n in $\operatorname{ind}_{m\to\infty} \mathscr{E}_{\{\omega\}}(X_n, m)$ such that $\rho_n^{-1}(U_n) \subseteq V$ where

$$\rho_n : \mathscr{E}_{\{\omega\}}(X) \to \operatorname{ind}_{m\to\infty} \mathscr{E}_{\{\omega\}}(X_n, m), \qquad f \mapsto f_{|X_n}.$$

The continuous map

$$\tilde{\rho}_{n+1} : \mathscr{E}_{\{\omega\}}(X_{n+1}) \to \operatorname{ind}_{m\to\infty} \mathscr{E}_{\{\omega\}}(X_n, m), \qquad f \mapsto f_{|X_n},$$

has the property that $\tilde{\rho}_{n+1}^{-1}(U_n)$ is a zero neighborhood in $\mathscr{E}_{\{\omega\}}(X_{n+1})$ for which

$$\tilde{\rho}_{n+1}^{-1}(U_n) \cap r^{X_{n+1}}_X(\mathscr{E}_{\{\omega\}}(X)) = r^{X_{n+1}}_X(\rho_n^{-1}(U_n)) \subseteq r^{X_{n+1}}_X(V).$$

For $k \in \mathbb{N}_0$, let

$$\pi_k : \operatorname{proj}_{\leftarrow n}(\mathscr{E}_{\{\omega\}}(X_n), r^{X_n}_{X_{n+1}}) \to \mathscr{E}_{\{\omega\}}(X_k), \qquad (f_n)_{n\in\mathbb{N}_0} \mapsto f_k,$$

so that

$$\begin{aligned} i(V) = \pi_{n+1}^{-1}(r^{X_{n+1}}_X(V)) &\supseteq \pi_{n+1}^{-1}\big(\tilde{\rho}_{n+1}^{-1}(U_n) \cap r^{X_{n+1}}_X(\mathscr{E}_{\{\omega\}}(X))\big) \\ &= \pi_{n+1}^{-1}(\tilde{\rho}_{n+1}^{-1}(U_n)). \end{aligned}$$

Since $\tilde{\rho}_{n+1}^{-1}(U_n)$ is a zero neighborhood in $\mathscr{E}_{\{\omega\}}(X_{n+1})$, the above inclusion implies that $i(V)$ is a zero neigborhood in $\operatorname{proj}_{\leftarrow n}(\mathscr{E}_{\{\omega\}}(X_n), r^{X_n}_{X_{n+1}})$, so that i is open and the sheaf $\mathscr{E}_{\{\omega\}}$ satisfies $(\mathscr{F}1)$.

Properties $(\mathscr{F}2)$–$(\mathscr{F}4)$ of $\mathscr{E}_{\{\omega\}}$ follow again from the fact that $\mathscr{E}_{\{\omega\}}(X)$ is closed under differentiation and contains all polynomials. ∎

Since for arbitrary weight functions ω the spaces $\mathscr{E}_{(\omega)}(X)$ and $\mathscr{E}_{\{\omega\}}(X)$ are locally convex algebras [BrMeTa, Proposition 4.4] it follows that $C_{w,\psi}$ is well-defined on $\mathscr{E}_{(\omega)}(X)$ resp. $\mathscr{E}_{\{\omega\}}(X)$ whenever the weight w belongs to the ultradifferentiable class, and additionally composition with ψ defines a continuous linear operator on $\mathscr{E}_{(\omega)}(X)$, resp. $\mathscr{E}_{\{\omega\}}(X)$. For non-quasianalytic weight functions ω this has been characterized by Fernández and Galbis [FeGa], while Rainer and Schindl [RaSc1, RaSc2] extended this characterization to more general weight functions. For a weight function ω we define the following property:

(α_0) $\exists C>0, t_0>0\ \forall \lambda\geq 1, t\geq t_0: \quad \omega(\lambda t)\leq C\lambda\omega(t)$.

Property (α_0) characterizes when composition with a smooth function $\psi: X\rightarrow X$ with components ψ_j, $1\leq j\leq d$, all belonging to $\mathscr{E}_{\{\omega\}}(X)$ defines a continuous linear operator on $\mathscr{E}_{\{\omega\}}(X)$. If $\omega(t)=o(t)$ as $t\rightarrow\infty$, then (α_0) characterizes when composition with a smooth function $\psi: X\rightarrow X$ with components $\psi_j, 1\leq j\leq d$, all belonging to $\mathscr{E}_{(\omega)}(X)$ is a continuous linear operator on $\mathscr{E}_{(\omega)}(X)$. Thus, under these conditions, $C_{w,\psi}$ is a well-defined, continuous linear operator which then also acts locally on $\mathscr{E}_{(\omega)}(X)$ resp. $\mathscr{E}_{\{\omega\}}(X)$.

THEOREM 5.2. *Let ω be a weight function satisfying (α_0) and such that $\omega(t)=o(t)$ as $t\rightarrow\infty$. Moreover, let $X\subseteq\mathbb{R}^d$ be open, $w\in\mathscr{E}_{(\omega)}(X)$, and $\psi: X\rightarrow X$ be smooth such that $\psi_j\in\mathscr{E}_{(\omega)}(X)$ for all $1\leq j\leq d$.*

(a) *The following are equivalent:*
   (i) *$C_{w,\psi}$ is weakly mixing on $\mathscr{E}_{(\omega)}(X)$.*
   (ii) *$C_{w,\psi}$ is hypercyclic on $\mathscr{E}_{(\omega)}(X)$.*
   (iii) *w has no zeros, ψ is injective, run-away, and* $\det J\psi(x)\neq 0$ *for all $x\in X$.*

(b) *The following are equivalent:*
   (i) *$C_{w,\psi}$ is mixing on $\mathscr{E}_{(\omega)}(X)$.*
   (ii) *w has no zeros, ψ is injective, strong run-away, and* $\det J\psi(x)\neq 0$ *for all $x\in X$.*

Proof. By a result due to Heinrich and Meise [HeMe, Proposition 3.2], $\mathscr{H}(\mathbb{C}^d)$ is dense in $\mathscr{E}_{(\omega)}(X)$. In particular, the (holomorphic) polynomials are dense in $\mathscr{E}_{(\omega)}(X)$, implying that the latter Fréchet space is separable. Because the polynomials are contained in $\mathscr{E}_{(\omega)}(\mathbb{R}^d)$ it follows that the latter space is dense in $C(\mathbb{R}^d)$. By Theorem 3.9 it thus follows that (i) implies (ii) and that (ii) implies (iii) in part (a).

If (a)(iii) is satisfied, it follows from the hypotheses on ω and [RaSc2, Theorem 4] that $1/w\in\mathscr{E}_{(\omega)}(X)$ and that for any $m\in\mathbb{N}$ the components of the smooth function $(\psi^m)^{-1}: X\rightarrow X$ belong to $\mathscr{E}_{(\omega)}(X)$. Therefore, applying again [RaSc2, Theorem 4] it follows that for every open subset Y of $\psi^m(X)$ and any $f\in\mathscr{E}_{(\omega)}((\psi^m)^{-1}(Y))$ the function

$$\tilde{f}: Y\rightarrow\mathbb{K}, y\mapsto\left(\frac{f}{\prod_{j=0}^{m-1}w(\psi^j(\cdot))}\right)\circ(\psi^m)^{-1}(y)$$

belongs to $\mathscr{E}_{(\omega)}(Y)$. As detailed in Remark 3.10(iii), this implies that condition (i)(a) of Theorem 3.9 is satisfied. Moreover, because ψ is run-away and $\det J\psi(x) \neq 0$ for every $x \in X$, conditions (i)(b1, b2) of Theorem 3.9 are fulfilled for any open, relatively compact exhaustion $(X_n)_{n\in\mathbb{N}}$ of X. Finally, applying [HeMe, Proposition 3.2] once more it follows in particular that (i)(b3) of Theorem 3.9 is satisfied, too, for an arbitrary open, relatively compact, exhaustion $(X_n)_{n\in\mathbb{N}}$ of X. Hence, by Theorem 3.9, (iii) implies (i) in part (a).

Mutatis mutandis, part (b) of the theorem is again proved by applying Theorem 3.11 instead of Theorem 3.9. ∎

THEOREM 5.3. *Let ω be a weight function satisfying (α_0). Moreover, let $X \subseteq \mathbb{R}^d$ be open, $w \in \mathscr{E}_{\{\omega\}}(X)$, and $\psi : X \to X$ be smooth such that $\psi_j \in \mathscr{E}_{\{\omega\}}(X)$ for all $1 \leq j \leq d$.*

(a) *The following are equivalent:*
   (i) *$C_{w,\psi}$ is weakly mixing on $\mathscr{E}_{\{\omega\}}(X)$.*
   (ii) *$C_{w,\psi}$ is transitive on $\mathscr{E}_{\{\omega\}}(X)$.*
   (iii) *w has no zeros, ψ is injective, run-away, and $\det J\psi(x) \neq 0$ for all $x \in X$.*

(b) *The following are equivalent:*
   (i) *$C_{w,\psi}$ is mixing on $\mathscr{E}_{\{\omega\}}(X)$.*
   (ii) *w has no zeros, ψ is injective, strong run-away, and $\det J\psi(x) \neq 0$ for all $x \in X$.*

Proof. We first prove (a). Clearly, (i) implies (ii), and since the polynomials are contained in $\mathscr{E}_{\{\omega\}}(\mathbb{R}^d)$, the latter space is dense in $C(\mathbb{R}^d)$, so that by Theorem 3.9, (iii) follows from (ii). If (iii) is satisfied, it follows as in the proof of Theorem 5.2 from [RaSc2, Theorem 3] and Remark 3.10(iii) that condition (i)(a) of Theorem 3.9 is fulfilled. Condition (i)(b3) in Theorem 3.9 is satisfied for any open, relatively compact exhaustion $(X_n)_{n\in\mathbb{N}}$ because by [HeMe, Proposition 3.2], $\mathscr{H}(\mathbb{C}^d)$ is dense in $\mathscr{E}_{\{\omega\}}(Y)$ for every open subset Y of $\mathbb{R}^d$. From the run-away property and the injectivity of ψ together with $\det J(x) \neq 0$ for all $x \in X$ it follows that (i)(b1, b2) of Theorem 3.9 are satisfied, too, so that (i) follows.

The proof of (b) is once more a straightforward modification of the proof of (a) involving Theorem 3.11. ∎

Since for the weight function $\omega(t) = t$ the spaces $\mathscr{E}_{\{\omega\}}(X)$ and $\mathscr{A}(X)$ coincide as locally convex spaces it follows that Corollary 4.3 is a special case of Theorem 5.3.

6. Kernels of elliptic differential operators. In this section we apply the results from Section 3 to weighted composition operators defined on kernels of elliptic partial differential operators. The special case of the

Cauchy–Riemann operator will give the space of holomorphic functions of a single variable equipped with the compact-open topology. In this context dynamical properties of (even a sequence of) unweighted composition operators have been studied by Bernal-González and Montes-Rodríguez [BeMo] and Große-Erdmann and Mortini [GEMo]. For dynamical properties of weighted composition operators on the Fréchet space of holomorphic functions see also [YoRe] and [Bes].

The special case of the Laplace operator gives the space of harmonic functions endowed with the compact-open topology, where dynamical properties of special unweighted composition operators have been studied for example in [Dz], [Ar]. The results in this section complement those from [CaMu], [KaNi], and [KaNiRe] where hypercyclicity of special unweighted composition operators on spaces of zero solutions to linear partial differential equations with constant coefficients is considered.

As explained in Example 2.3(v), for a non-constant $P \in \mathbb{C}[X_1, \ldots, X_d]$ with $d \geq 2$ and an open subset $X \subseteq \mathbb{R}^d$ we define

$$C_P^\infty(X) := \{u \in C^\infty(X);\ P(\partial)u = 0 \text{ in } X\},$$

where for $P(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$ with $a_{\alpha_0} \neq 0$ for some multiindex $\alpha_0 \in \mathbb{N}_0^d$ with $|\alpha_0|$ $(= \alpha_1 + \cdots + \alpha_d) = m$ we define

$$\forall u \in C^\infty(X), x \in X: \quad P(\partial)u(x) = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha u(x).$$

As a closed subspace of the separable nuclear Fréchet space $C^\infty(X)$, the space $C_P^\infty(X)$ is then again a separable nuclear Fréchet space. For hypoelliptic polynomials P (by definition) for every open $X \subseteq \mathbb{R}^d$ the spaces $C_P^\infty(X)$ and

$$\mathscr{D}'_P(X) := \{u \in \mathscr{D}'(X);\ P(\partial)u = 0 \text{ in } X\}$$

coincide (that is, every distribution u on X which satisfies $P(\partial)u = 0$ in X is already a smooth function). By a result of Malgrange (see e.g. [Tr, Theorem 52.1]) the spaces $C_P^\infty(X)$ and $\mathscr{D}'_P(X)$ also coincide as locally convex spaces when the latter is endowed with the relative topology inherited from $\mathscr{D}'(X)$ equipped with the strong dual topology as the topological dual of $\mathscr{D}(X)$. This implies in particular that for hypoelliptic polynomials the compact-open topology on $C_P^\infty(X)$ and the relative topology inherited from $C^\infty(X)$ coincide. Therefore, for hypoelliptic polynomials P the space $C_P^\infty(X)$ endowed with the compact-open topology is a separable (nuclear) Fréchet space for every open $X \subseteq \mathbb{R}^d$.

As already mentioned in Example 2.3(v), C_P^∞ defines a sheaf on $\mathbb{R}^d$ which satisfies ($\mathscr{F}1$) and ($\mathscr{F}2$) but generally ($\mathscr{F}3$) need not hold. However, the next proposition shows that for hypoelliptic polynomials P both ($\mathscr{F}3$) and ($\mathscr{F}4$) hold for C_P^∞.

PROPOSITION 6.1. *Let $d \geq 2$ and let $P \in \mathbb{C}[X_1, \ldots, X_d]$ be hypoelliptic. The sheaf C_P^∞ satisfies both* ($\mathscr{F}3$) *and* ($\mathscr{F}4$).

Proof. Fix distinct $x, y \in \mathbb{R}^d$. By renumbering the coordinates we can assume that $x_d - y_d \neq 0$. For $\xi' \in \mathbb{R}^{d-1}$ we denote by $\lambda_1(\xi'), \ldots, \lambda_{l(\xi')}(\xi') \in \mathbb{C}$ the pairwise distinct roots of the polynomial

$$\mathbb{C} \to \mathbb{C}, \qquad z \mapsto P(\xi', z),$$

ordered in such a way that $(\operatorname{Im} \lambda_j(\xi'))_{1 \leq j \leq l(\xi')}$ is increasing and $\operatorname{Re} \lambda_j(\xi') < \operatorname{Re} \lambda_{j+1}(\xi')$ whenever $\operatorname{Im} \lambda_j(\xi') = \operatorname{Im} \lambda_{j+1}(\xi')$.

Then the mapping

$$\operatorname{Im} \lambda_1 : \mathbb{R}^{d-1} \to \mathbb{R}, \qquad \xi' \mapsto \operatorname{Im} \lambda_1(\xi'),$$

is continuous. Indeed, fix $\xi_0' \in \mathbb{R}^{d-1}$ and let m_j be the multiplicity of $\lambda_j(\xi_0')$. Let $\varepsilon > 0$. We can assume that ε is so small that for every j we have $\operatorname{Im} \lambda_j(\xi_0') + \varepsilon < \operatorname{Im} \lambda_{j+1}(\xi_0') - \varepsilon$ if $\operatorname{Im} \lambda_j(\xi_0') < \operatorname{Im} \lambda_{j+1}(\xi_0')$, and $\operatorname{Re} \lambda_j(\xi_0') + \varepsilon < \operatorname{Re} \lambda_{j+1}(\xi_0') - \varepsilon$ if $\operatorname{Im} \lambda_j(\xi_0') = \operatorname{Im} \lambda_{j+1}(\xi_0')$. Then $B(\lambda_j(\xi_0'), \varepsilon) \cap B(\lambda_k(\xi_0'), \varepsilon) = \emptyset$ for every $1 \leq j, k \leq l(\xi_0')$, $j \neq k$. For any $1 \leq j \leq l(\xi_0')$ we have by Taylor's Theorem for every $\xi' \in \mathbb{R}^{d-1}$ with $|\xi' - \xi_0'| < 1$,

$$\begin{aligned}
&\forall z \in \mathbb{C},\ |z - \lambda_j(\xi_0')| = \varepsilon : \\
&|P(\xi', z) - P(\xi_0', z)| = \Big| \sum_{\alpha \neq 0} P^{(\alpha)}((\xi_0', 0) + z e_d) \frac{(\xi' - \xi_0', 0)^\alpha}{\alpha!} \Big| \\
&\qquad \leq \sum_{\alpha \neq 0} |P^{(\alpha)}((\xi_0', 0) + z e_d)| \frac{|\xi' - \xi_0'|^{|\alpha|}}{\alpha!} \\
&\qquad \leq |\xi' - \xi_0'| \sup_{\zeta \in \mathbb{C},\, |\zeta - \lambda_j(\xi_0')| = \varepsilon} \sum_{\alpha \neq 0} |P^{(\alpha)}((\xi_0', 0) + \zeta e_d)| \frac{1}{\alpha!}.
\end{aligned}$$

Thus, if $|\xi' - \xi_0'|$ is sufficiently small, the right hand side of the above inequality is less than

$$\inf\{|P(\xi_0', \zeta)|;\ \zeta \in \mathbb{C}, |\zeta - \lambda_j(\xi_0')| = \varepsilon\} \quad (\leq |P(\xi_0', z)|\ (|z - \lambda_j(\xi_0')| = \varepsilon))$$

for any j. Hence it follows from Rouché's Theorem for ξ' sufficiently close to ξ_0', say $|\xi' - \xi_0'| < \delta$, that $P(\xi', \cdot)$ has exactly m_j roots $z_{j,1}^{\xi'}, \ldots, z_{j,m_j}^{\xi'}$ in $B(\lambda_j(\xi_0'), \varepsilon)$ for each $1 \leq j \leq l(\xi_0')$.

Now set $k := \max\{1 \leq j \leq l(\xi_0');\ \operatorname{Im} \lambda_1(\xi_0') = \operatorname{Im} \lambda_j(\xi_0')\}$. Then for any $1 \leq j \leq k, 1 \leq r \leq m_j$ we have

$$|\operatorname{Im} z_{j,r}^{\xi'} - \operatorname{Im} \lambda_j(\xi_0')| < \varepsilon$$

if $|\xi' - \xi_0'| < \delta$, and according to our choice of ε we have

$$\max\{\operatorname{Im} z_{j,r}^{\xi'};\ 1 \le j \le k,\ 1 \le r \le m_j\} < \min\{\operatorname{Im} z_{j,r}^{\xi'};\ k+1 \le j \le k,\ 1 \le r \le m_j\}.$$

Therefore $\operatorname{Im}\lambda_1(\xi'), \ldots, \operatorname{Im}\lambda_{m_1}(\xi'), \operatorname{Im}\lambda_{m_1+1}(\xi'), \ldots, \operatorname{Im}\lambda_{m_k}(\xi')$ all belong to

$$(\operatorname{Im}\lambda_1(\xi_0') - \varepsilon, \operatorname{Im}\lambda_1(\xi_0') + \varepsilon) = \cdots = (\operatorname{Im}\lambda_k(\xi_0') - \varepsilon, \operatorname{Im}\lambda_k(\xi_0') + \varepsilon),$$

so that for all $1 \le j \le m_k$ we have $|\operatorname{Im}\lambda_j(\xi') - \operatorname{Im}\lambda_1(\xi_0')| < \varepsilon$ when $|\xi' - \xi_0'| < \delta$. In particular, $|\lambda_1(\xi') - \lambda_1(\xi_0')| < \varepsilon$ whenever $|\xi' - \xi_0'| < \delta$, which gives the continuity of $\operatorname{Im}\lambda_1$.

Denote $V(P) := \{\zeta \in \mathbb{C}^d;\ P(\zeta) = 0\}$. Then $(\xi', \lambda_1(\xi')) \in V(P)$ for each $\xi' \in \mathbb{R}^{d-1}$ and therefore

$$\forall \xi' \in \mathbb{R}^{d-1}: \quad \operatorname{dist}((\xi', \operatorname{Re}\lambda_1(\xi')), V(P)) \le |\operatorname{Im}\lambda_1(\xi')|. \tag{6}$$

Since P is hypoelliptic we have

$$\lim_{x \in \mathbb{R}^d,\, |x| \to \infty} \operatorname{dist}(x, V(P)) = \infty$$

(see [Hö1, Theorem 11.1.3]), which combined with (6) yields

$$\lim_{|\xi'| \to \infty} |\operatorname{Im}\lambda_1(\xi')| = \infty. \tag{7}$$

This implies in particular that there are $\xi', \eta' \in \mathbb{R}^{d-1}$ for which

$$\big\langle \operatorname{Im}\big((\xi', \lambda_1(\xi')) - (\eta', \lambda_1(\eta'))\big), x - y\big\rangle = \operatorname{Im}(\lambda_1(\xi') - \lambda_1(\eta'))(x_d - y_d) \notin \{2\pi k;\ k \in \mathbb{Z}\}.$$

Thus, there are $\zeta_1, \zeta_2 \in V(P)$ such that $\langle \operatorname{Im}(\zeta_1 - \zeta_2), x - y\rangle$ is not an integer multiple of 2π. Set $\langle \eta, v\rangle = \sum_{j=1}^d \eta_j v_j$ for $\eta, v \in \mathbb{C}^d$. The function

$$g : \mathbb{R}^d \to \mathbb{C}, \quad g(w) := \exp(\langle \zeta_1, x\rangle + \langle \zeta_2, w\rangle) - \exp(\langle \zeta_2, x\rangle + \langle \zeta_1, w\rangle),$$

satisfies $g \in C_P^\infty(\mathbb{R}^d)$, $g(x) = 0$, and

$$g(y) = \exp(\langle \zeta_1, y\rangle + \langle \zeta_2, x\rangle)\big(\exp(\langle \zeta_1 - \zeta_2, x - y\rangle) - 1\big) \ne 0,$$

which implies the existence of $f \in C_P^\infty(\mathbb{R}^d)$ with $f(x) = 0$ and $f(y) = 1$. Hence, $(\mathscr{F}3)$ is satisfied.

To verify $(\mathscr{F}4)$, let $X \subseteq \mathbb{R}^d$ be open. We first observe that $-\partial_j \delta_x$, $1 \le j \le d$, is a continuous linear functional on $C_P^\infty(X)$. Now, let $h \in \mathbb{R}^d \setminus \{0\}$ and $\lambda \in \mathbb{K}$. By renumbering the coordinates if necessary we may assume that $h_d \ne 0$. By (7) it follows that

$$\exists \zeta \in V(P): \quad \langle h, \operatorname{Im}\zeta\rangle - \operatorname{Im}\lambda \ne 0. \tag{8}$$

Because $e_\zeta \in C_P^\infty(X)$, where $e_\zeta(x) := \exp(\langle \zeta, x\rangle)$,

$$\sum_{j=1}^d h_j \partial_j e_\zeta(x) - \lambda e_\zeta(x) = \big(\langle h, \operatorname{Re}\zeta\rangle - \operatorname{Re}\lambda + i(\langle h, \operatorname{Im}\zeta\rangle - \operatorname{Im}\lambda)\big) e_\zeta(x)$$

where the factor

$$\big(\langle h, \operatorname{Re}\zeta\rangle - \operatorname{Re}\lambda + i(\langle h, \operatorname{Im}\zeta\rangle - \operatorname{Im}\lambda)\big)$$

does not vanish by (8). Therefore the continuous linear functional

$$u \mapsto \sum_{j=1}^d h_j \partial_j u - \lambda u$$

on $C_P^\infty(X)$ does not vanish identically, so that ($\mathscr{F}4$) is fulfilled. ■

Elliptic polynomials will be of particular interest for us. Recall that a polynomial $P \in \mathbb{C}[X_1, \ldots, X_d]$, $P(\xi) = \sum_{|\alpha|\le m} a_\alpha \xi^\alpha$, is called *elliptic* if

$$\forall \xi \in \mathbb{R}^d \setminus \{0\}: \quad P_m(\xi) \neq 0,$$

where $P_m(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha$ denotes the principal part of P. As is well-known, elliptic polynomials are hypoelliptic (see e.g. [Hö1, Theorem 11.1.10]). In particular, identifying $\mathbb{C}$ as usual with $\mathbb{R}^2$, and choosing for $P \in \mathbb{C}[X_1, X_2]$ the polynomial $P(\xi_1, \xi_2) = \frac{1}{2}(\xi_1 + i\xi_2)$ gives the Cauchy–Riemann operator $\partial_{\bar z}$ and we know that $C_P^\infty(X) = \mathscr{H}(X)$ holds as locally convex spaces for any open $X \subseteq \mathbb{C}$ so that the sheaf of holomorphic functions (equipped with the compact-open topology) on open subsets of $\mathbb{C}$ is a special case.

Arguably the most prominent elliptic differential operator, apart from the Cauchy–Riemann operator, is the Laplace operator. Thus, the sheaf of harmonic functions (equipped with the compact-open topology) on open subsets of $\mathbb{R}^d$ is also a special case of the sheaves C_P^∞.

We are now going to characterize when for an elliptic polynomial P and an open $X \subseteq \mathbb{R}^d$ a well-defined weighted composition operator $C_{w,\psi}$ on $C_P^\infty(X)$ is weakly mixing. As follows in particular from the results of [GEMo], an unweighted composition operator cannot be hypercyclic on $\mathscr{H}(X)$ if X is a finitely connected but not simply connected domain. Thus, the special case of the Cauchy–Riemann operator shows that topological properties of X have to be taken into account.

THEOREM 6.2. *Let P be an elliptic polynomial and let $X \subseteq \mathbb{R}^d$ be open and homeomorphic to $\mathbb{R}^d$. Moreover, let $w: X \to \mathbb{C}$ and $\psi: X \to X$ be smooth such that $C_{w,\psi}$ is well-defined on $C_P^\infty(X)$ and acts locally on $C_P^\infty(X)$.*

(a) *The following are equivalent:*

  (i) *$C_{w,\psi}$ is weakly mixing on $C_P^\infty(X)$.*

(ii) *$C_{w,\psi}$ has dense range, w has no zeros, and ψ is injective and run-away.*

(iii) *w has no zeros, ψ is injective and run-away, and for each $m \in \mathbb{N}_0$ and every open, relatively compact $Y \subseteq \psi^m(X)$,*
$$r_X^{(\psi^m)^{-1}(Y)}(C_P^\infty(X)) \subseteq \overline{(C_{w,\psi,(\psi^{m-1})^{-1}(Y)} \circ \cdots \circ C_{w,\psi,Y})(C_P^\infty(Y))},$$
where the closure is taken in $C_P^\infty((\psi^m)^{-1}(Y))$.

Moreover, $\det J\psi(x) \neq 0$ *for all $x \in X$ can be added to* (ii) *and* (iii). *If additionally $|w(x)| \leq 1$ for all $x \in X$ then the above are equivalent to*

(iv) *$C_{w,\psi}$ is hypercyclic on $C_P^\infty(X)$.*

(b) *The following are equivalent:*

(i) *$C_{w,\psi}$ is mixing on $C_P^\infty(X)$.*

(ii) *$C_{w,\psi}$ has dense range, w has no zeros, and ψ is injective and strong run-away.*

(iii) *w has no zeros, ψ is injective and strong run-away, and for each $m \in \mathbb{N}_0$ and every open, relatively compact $Y \subseteq \psi^m(X)$,*
$$r_X^{(\psi^m)^{-1}(Y)}(C_P^\infty(X)) \subseteq \overline{(C_{w,\psi,(\psi^{m-1})^{-1}(Y)} \circ \cdots \circ C_{w,\psi,Y})(C_P^\infty(Y))},$$
where the closure is taken in $C_P^\infty((\psi^m)^{-1}(Y))$.

Moreover, $\det J\psi(x) \neq 0$ *for all $x \in X$ can be added to* (ii) *and* (iii).

For the proof of Theorem 6.2 some preparations have to be made. First, let us mention that in the case of $\mathbb{C} = \mathbb{R}^2$ it follows from the Riemann Mapping Theorem that every simply connected, connected, open $X \subseteq \mathbb{C}$ different from $\mathbb{C}$ is in particular homeomorphic to the open unit disc in $\mathbb{C}$ which itself is homeomorphic to $\mathbb{C}$. Thus, for $d = 2$ the topological hypothesis on X in Theorem 6.2 means precisely that X is a simply connected domain.

In order to prove Theorem 6.2 we need the following version of the celebrated Jordan–Brouwer Separation Theorem that can be found in [May, Satz 5.23]. Since this reference is in German and we could not find a different reference, we include a proof (different from the one in [May]) for the reader's convenience.

THEOREM 6.3 (Version of Jordan–Brouwer Separation Theorem). *Let K_1 and K_2 be homeomorphic compact subsets of $\mathbb{R}^d$. Then $\mathbb{R}^d \setminus K_1$ and $\mathbb{R}^d \setminus K_2$ have the same number of connected components.*

Proof. For a topological space X we denote as usual the nth reduced homology group, respectively cohomology group, with coefficients in $\mathbb{Z}$ by $\tilde{H}_n(X, \mathbb{Z})$ respectively $\tilde{H}^n(X, \mathbb{Z})$. Moreover, let S^d be the unit sphere in $\mathbb{R}^{d+1}$, $N := (1, 0, \ldots, 0) \in S^d$ be the "north pole", and $\phi : \mathbb{R}^d \to S^d \setminus \{N\}$ be a homeomorphism. Since K_1 and K_2 are homeomorphic, the compact subsets

$\{N\}\,\dot\cup\,\phi(K_1)$ and $\{N\}\,\dot\cup\,\phi(K_2)$ of S^d are homeomorphic. Thus, the groups $\tilde{H}^{d-1}(\{N\}\dot\cup\phi(K_1),\mathbb{Z})$ and $\tilde{H}^{d-1}(\{N\}\dot\cup\phi(K_2),\mathbb{Z})$ are isomorphic. By Alexander Duality (see [Ha, Theorem 3.44]) the groups $\tilde{H}_0(S^d\setminus(\{N\}\,\dot\cup\,\phi(K_1)),\mathbb{Z})$ and $\tilde{H}_0(S^d\setminus(\{N\}\dot\cup\phi(K_2)),\mathbb{Z})$ are isomorphic and thus so are $\tilde{H}_0(\mathbb{R}^d\setminus K_1,\mathbb{Z})$ and $\tilde{H}_0(\mathbb{R}^d\setminus K_2,\mathbb{Z})$. Hence, $\tilde{H}_0(\mathbb{R}^d\setminus K_1,\mathbb{Z})\oplus\mathbb{Z}$ and $\tilde{H}_0(\mathbb{R}^d\setminus K_2,\mathbb{Z})\oplus\mathbb{Z}$ are isomorphic as well. Since $\tilde{H}_0(X,\mathbb{Z})\oplus\mathbb{Z}$ and $H_0(X,\mathbb{Z})$, the homology group of degree zero of X, are isomorphic (see e.g. [Ha, p. 110]), and since $H_0(X,\mathbb{Z})$ is isomorphic to $\bigoplus_{\alpha\in C(X)}\mathbb{Z}$, where $C(X)$ is the set of all pathwise connected components of X, it follows that $\mathbb{R}^d\setminus K_1$ and $\mathbb{R}^d\setminus K_2$ have the same number of pathwise connected components. Since $\mathbb{R}^d\setminus K_j$ are open in $\mathbb{R}^d$ and thus locally pathwise connected it follows that $\mathbb{R}^d\setminus K_1$ and $\mathbb{R}^d\setminus K_2$ have indeed the same number of connected components. ∎

Recall that $B(x,\varepsilon)$, resp. $B[x,\varepsilon]$, denotes an open, resp. closed, ball.

Proposition 6.4. *For every continuous and injective $\psi:\mathbb{R}^d\to\mathbb{R}^d$ the following hold:*

(i) $\forall n\in\mathbb{N}:\overline{\mathbb{R}^d\setminus\psi(B[0,n])}=\mathbb{R}^d\setminus\psi(B(0,n))$, $\overline{\mathbb{R}^d\setminus B[0,n]}=\mathbb{R}^d\setminus B(0,n)$.

(ii) *If $d\geq 2$ then $\mathbb{R}^d\setminus\psi(B[0,n])$ and $\mathbb{R}^d\setminus\psi(B(0,n))$ are connected for every $n\in\mathbb{N}$.*

(iii) *If $d\geq 2$ and $n\in\mathbb{N}$ then $\mathbb{R}^d\setminus(\psi(B(0,n))\cup B(0,n))$ is connected whenever $\psi(B[0,n])\cap B[0,n]=\emptyset$.*

Proof. Denoting the interior of $A\subseteq\mathbb{R}^d$ by $\mathrm{int}(A)$ we have $\mathbb{R}^d\setminus\mathrm{int}(A)=\overline{\mathbb{R}^d\setminus A}$ for any $A\subseteq\mathbb{R}^d$. Thus, $\overline{\mathbb{R}^d\setminus\psi(B[0,n])}=\mathbb{R}^d\setminus\mathrm{int}(\psi(B[0,n]))$. Since ψ is continuous and injective it follows from Brouwer's Invariance of Domain Theorem that $\psi(B(0,n))$ is open in $\mathbb{R}^d$. Thus, $\psi(B(0,n))\subseteq\mathrm{int}(\psi(B[0,n]))$. On the other hand, for x in the interior of $\psi(B[0,n])$ there is $\delta>0$ such that

$$\psi^{-1}(B(x,\delta))\subseteq\psi^{-1}(\psi(B[0,n]))=B[0,n],$$

where we have used the injectivity of ψ. Since $\psi^{-1}(B(x,\delta))$ is open in $\mathbb{R}^d$, we conclude that $\psi^{-1}(B(x,\delta))\subseteq B(0,n)$. From

$$B(x,\delta)\subseteq\psi(B[0,n])\subseteq\psi(\mathbb{R}^d)$$

and the injectivity of ψ we get

$$B(x,\delta)=\psi(\psi^{-1}(B(x,\delta)))\subseteq\psi(B(0,n)).$$

Since $x\in\mathrm{int}(\psi(B[0,n]))$ was chosen arbitrarily it follows $\mathrm{int}(\psi(B[0,n]))\subseteq\psi(B(0,n))$ so that $\mathrm{int}(\psi(B[0,n]))=\psi(B(0,n))$, which proves (i).

In order to prove (ii), for $n\in\mathbb{N}$ we define

$$\psi_n:B[0,n]\to\psi(B[0,n]),\qquad x\mapsto\psi(x),$$

which is a continuous bijection, thus a homeomorphism by compactness. $\mathbb{R}^d\setminus B[0,n]$ is connected because $d\geq 2$ so that by Theorem 6.3 the same is true for $\mathbb{R}^d \setminus \psi(B[0,n])$. Therefore, by (i), the set

$$\overline{\mathbb{R}^d \setminus \psi(B[0,n])} = \mathbb{R}^d \setminus \psi(B(0,n))$$

is connected, too, which proves (ii).

In order to show (iii), we first observe that

$$\overline{\mathbb{R}^d \setminus (\psi(B[0,n]) \cup B(0,n))} = \mathbb{R}^d \setminus \mathrm{int}(\psi(B[0,n]) \cup B(0,n)).$$

Clearly,

$$\mathrm{int}(\psi(B[0,n]) \cup B(0,n)) \supseteq \mathrm{int}(\psi(B[0,n])) \cup B(0,n),$$

and because $\psi(B[0,n])$ and $B[0,n]$ are disjoint closed sets we also have

$$\mathrm{int}(\psi(B[0,n]) \cup B(0,n)) \subseteq \mathrm{int}(\psi(B[0,n])) \cup B(0,n),$$

which combined with $\mathrm{int}(\psi(B[0,n])) = \psi(B(0,n))$ gives

$$\overline{\mathbb{R}^d \setminus (\psi(B[0,n]) \cup B(0,n))} = \mathbb{R}^d \setminus (\psi(B(0,n)) \cup B(0,n)).$$

Because the closure of a connected set is connected, it suffices to show the connectedness of the set $\mathbb{R}^d \setminus (\psi(B[0,n]) \cup B(0,n))$. Let x, y be in this set. By (ii), $\mathbb{R}^d \setminus \psi(B[0,n])$ is connected. Because open, connected subsets of $\mathbb{R}^d$ are pathwise connected, there is a continuous $\gamma : [0,1] \to \mathbb{R}^d \setminus \psi(B[0,n])$ with $\gamma(0) = x$ and $\gamma(1) = y$. If $\gamma([0,1])$ does not intersect $B(0,n)$, we are done. Otherwise let

$$t_0 := \inf\{t \in [0,1];\, \gamma(t) \in B(0,n)\}, \qquad t_1 := \sup\{t \in [0,1];\, \gamma(t) \in B(0,n)\}.$$

Then $0 < t_0 \le t_1 < 1$ and $\gamma(t_j) \in \partial B(0,n)$, $j = 0,1$. Since $d \ge 2$, $\partial B(0,n)$ is pathwise connected, so there is a continuous $\alpha : [t_0, t_1] \to \partial B(0,n)$ such that $\alpha(t_j) = \gamma(t_j)$, $j = 0,1$. Then

$$\tilde{\gamma} : [0,1] \to \mathbb{R}^d \setminus (\psi(B[0,n])) \cup B(0,n)), \qquad t \mapsto \begin{cases} \gamma(t), & t \notin [t_0, t_1], \\ \alpha(t), & t \in [t_0, t_1], \end{cases}$$

is a well-defined continuous mapping with $\tilde{\gamma}(0) = x$ and $\tilde{\gamma}(1) = y$, which proves (iii). ■

Proposition 6.5. *Let $d \ge 2$ and $X \subseteq \mathbb{R}^d$ be open and homeomorphic to $\mathbb{R}^d$ and let $\psi : X \to X$ be continuous, injective, and run-away. Then there is a relatively compact-open exhaustion $(X_n)_{n\in\mathbb{N}}$ of X such that*

$$\forall n \in \mathbb{N}\ \exists m \in \mathbb{N} : \quad X_n \cap \psi^m(X_n) = \emptyset,$$

and if $m, n \in \mathbb{N}$ are such that X_n and $\psi^m(X_n)$ are disjoint, and $X \setminus (X_n \cup \psi^m(X_n)) = F \dot{\cup} K$ where F is (relatively) closed in X and $K \subseteq X$ is compact, then $K = \emptyset$.

Proof. We first assume that $X = \mathbb{R}^d$. From the hypothesis it follows that for each $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that $B(0,n)$ and $\psi^m(B(0,n))$ are disjoint. Applying Proposition 6.4 to ψ^m we find that $\mathbb{R}^d \setminus (B(0,n) \cup \psi^m(B(0,n)))$

is connected. In particular, for every closed set $F \subseteq \mathbb{R}^d$ and each compact subset K of $\mathbb{R}^d \setminus F$ we have

$$\mathbb{R}^d \setminus (B(0,n) \cup \psi^m(B(0,n))) = F \,\dot{\cup}\, K \Rightarrow K = \emptyset.$$

Thus, for $X = \mathbb{R}^d$ we can choose $X_n := B(0,n)$, $n \in \mathbb{N}$.

Now let $X \subseteq \mathbb{R}^d$ be an arbitrary open subset homeomorphic to $\mathbb{R}^d$ via $\Phi : X \to \mathbb{R}^d$. Then $\Phi \circ \psi \circ \Phi^{-1}$ is a continuous, injective mapping on $\mathbb{R}^d$ with the run-away property. For $n \in \mathbb{N}$ let $X_n := \Phi^{-1}(B(0,n))$ so that $(X_n)_{n\in\mathbb{N}}$ is an open, relatively compact exhaustion of X. Let $n, m \in \mathbb{N}$ be such that X_n and $\psi^m(X_n)$ are disjoint and let $F \subseteq X$ be relatively closed and $K \subseteq X$ be compact such that

$$F \,\dot{\cup}\, K = X \setminus (X_n \cup \psi^m(X_n)) = \Phi^{-1}\big(\mathbb{R}^d \setminus (B(0,n) \cup (\Phi \circ \psi \circ \Phi)^m(B(0,n)))\big).$$

Since $\Phi(F)$ is a closed subset of $\mathbb{R}^d$ and $\Phi(K)$ is compact it follows together with

$$\Phi(F) \,\dot{\cup}\, \Phi(K) = \mathbb{R}^d \setminus \big(B(0,n) \cup (\Phi \circ \psi \circ \Phi)^m(B(0,n))\big)$$

and the case of $X = \mathbb{R}^d$ applied to $\Phi \circ \psi \circ \Phi^{-1}$ that $\Phi(K) = \emptyset$, hence $K = \emptyset$. ∎

Proof of Theorem 6.2. Since P is elliptic, the sheaf C_P^∞ satisfies $(\mathscr{F}1)$–$(\mathscr{F}4)$ by Proposition 6.1. We can therefore invoke Theorem 3.9 in order to prove part (a). If (i) holds, i.e. $C_{w,\psi}$ is weakly mixing, then $C_{w,\psi}$ obviously has dense range and the rest of (ii) follows from Theorem 3.9. If (ii) holds, it follows from Proposition 3.8 that (iii) is true.

Next, if (iii) holds, condition (i)(a) of Theorem 3.9 is fulfilled. Let $(X_n)_{n\in\mathbb{N}}$ be the open, relatively compact exhaustion of X from Proposition 6.5. By the injectivity of ψ it follows from Brouwer's Invariance of Domain Theorem that (i)(b1) in Theorem 3.9 is satisfied, while (i)(b2) is satisfied since ψ is run-away. Fix $n \in \mathbb{N}$ and let $m \in \mathbb{N}$ be such that X_n and $\psi^m(X_n)$ are disjoint. It follows from Proposition 6.5 that it is not possible to decompose $X \setminus (X_n \cup \psi^m(X_n))$ into a relatively closed subset of X and a non-empty compact subset of K which are disjoint. Since P is elliptic, it follows from the Lax–Malgrange Theorem (see e.g. [Hö1, Theorem 4.4.5 combined with the remark preceding Corollary 4.4.4 resp. with Theorem 8.6.1] or [Na, Theorem 3.10.7]) that

$$\{u_{|X_n \cup \psi^m(X_n)};\ u \in C_P^\infty(X)\}$$

is dense in $C_P^\infty(X_n \cup \psi^m(X_n))$, i.e. $r_X^{X_n \cup \psi^m(X_n)}$ has dense range. Thus, conditions (i)(a, b) of Theorem 3.9 are satisfied, so by that theorem $C_{w,\psi}$ is weakly mixing. Thus (i)–(iii) are equivalent. If additionally $|w(x)| \leq 1$ and $C_{w,\psi}$ is hypercyclic it follows from Theorem 3.9 that (iii) holds. Since trivially (i) implies (iv), (a) is proved.

The proof of part (b) is mutatis mutandis a repetition of the above arguments with the reference to Theorem 3.9 replaced by a reference to Theorem 3.11. ∎

In the remainder of this section we are going to characterize the dynamics for weighted composition operators on eigenspaces of the Cauchy–Riemann operator and Laplace operator respectively, i.e. on C_P^∞ for the polynomial in $d=2$ variables $P(\xi)=\frac{1}{2}(\xi_1+i\xi_2)-\lambda$, resp. in d variables $P(\xi)=\sum_{j=1}^d \xi_j^2-\lambda$ where in both cases $\lambda\in\mathbb{C}$ is arbitrary. We begin our considerations for these special operators by determining explicitly the combinations of symbols and weights which yield well-defined weighted composition operators on $C_P^\infty(X)$.

Proposition 6.6. (a) *Let $\lambda\in\mathbb{C}$, $d=2$ and $P(\xi)=\frac{1}{2}(\xi_1+i\xi_2)-\lambda$. For $X\subseteq\mathbb{R}^2=\mathbb{C}$ open, and $w:X\to\mathbb{C}$ and $\psi:X\to X$ smooth, the following are equivalent:*

(i) *$C_{w,\psi}$ is well-defined on $C_P^\infty(X)$.*
(ii) *$w\partial_{\bar z}\psi=0$ and $P(\partial)w=-\lambda w\,\partial_{\bar z}\bar\psi$.*

Moreover, if $C_{w,\psi}$ is well-defined on $C_P^\infty(X)$ then for every $Y\subseteq X$ open and $f\in C^\infty(Y)$ we have

$$P(\partial)(C_{w,\psi}(f))=\partial_{\bar z}\bar\psi\,C_{w,\psi}(P(\partial)f).$$

(b) *Let $\lambda\in\mathbb{C}$, $d\geq 2$ and $P(\xi)=\sum_{j=1}^d\xi_j^2-\lambda$. For $X\subseteq\mathbb{R}^d$ open, and $w:X\to\mathbb{C}$ and $\psi:X\to X$ smooth, the following are equivalent:*

(i) *$C_{w,\psi}$ is well-defined on $C_P^\infty(X)$.*
(ii) *For every $1\leq j\neq k\leq d$ we have $w|\nabla\psi_j|^2=w|\nabla\psi_k|^2$, $w\langle\nabla\psi_j,\nabla\psi_k\rangle=0$, $w\Delta\psi_j+2\langle\nabla w,\nabla\psi_j\rangle=0$, and $P(\partial)w=-\lambda w|\nabla\psi_1|^2$.*

Moreover, if $C_{w,\psi}$ is well-defined on $C_P^\infty(X)$ then for every $Y\subseteq X$ open and $f\in C^\infty(Y)$ we have

$$P(\partial)(C_{w,\psi}(f))=|\nabla\psi_1|^2\,C_{w,\psi}(P(\partial)f).$$

Proof. We write $f(\psi)$ instead of $f\circ\psi$ to slightly simplify notation. To prove (a) it is straightforward to verify that for every $f\in C^\infty(X)$,

$$P(\partial)(w\cdot f(\psi))=(P(\partial)w)f(\psi)+w(\partial_z f)(\psi)\partial_{\bar z}\psi+w(\partial_{\bar z}f)(\psi)\partial_{\bar z}\bar\psi. \tag{9}$$

Now assume that (a)(i) holds. Inserting $f=e_\zeta$ with $\zeta=(2\lambda,0)\in\mathbb{C}^2$ into (9), since $e_\zeta\in C_P^\infty(X)$ (recall that $e_\zeta(x)=\exp(\sum_{j=1}^2\zeta_jx_j)$), we see that

$$0=\big(P(\partial)w+\lambda w(\partial_{\bar z}\psi+\partial_{\bar z}\bar\psi)\big)e_\zeta$$

so that

$$0=P(\partial)w+\lambda w(\partial_{\bar z}\psi+\partial_{\bar z}\bar\psi). \tag{10}$$

Likewise, we derive from (9) by inserting $f = e_\eta$ with $\eta = (0, -2i\lambda) \in \mathbb{C}^2$ that

$$0 = P(\partial)w - \lambda w(\partial_{\bar z}\psi - \partial_{\bar z}\bar\psi). \tag{11}$$

Subtracting (11) from (10) yields

$$0 = \lambda w \partial_{\bar z}\psi, \tag{12}$$

while adding (10) and (11) gives

$$0 = P(\partial)w + \lambda w \partial_{\bar z}\bar\psi. \tag{13}$$

For $\lambda = 0$ we evaluate (9) for $f(x) = x_1 + ix_2$, which gives $w\partial_{\bar z}\psi = 0$. If $\lambda \neq 0$ we have $w\partial_{\bar z}\psi = 0$, too, by (12), showing one half of (ii). Additionally, evaluating (9) for an arbitrary $f \in C_P^\infty(X)$ gives

$$\begin{aligned} \forall f \in C_P^\infty(X): \quad 0 &= (P(\partial)w)f(\psi) + w(\partial_z f)(\psi)\partial_{\bar z}\psi + w(\partial_{\bar z} f)(\psi)\partial_{\bar z}\bar\psi \\ &= (P(\partial)w)f(\psi) + \lambda w f(\psi)\partial_{\bar z}\bar\psi \\ &= \big(\partial_{\bar z} w - (1 - \partial_{\bar z}\bar\psi)\lambda w\big) f(\psi). \end{aligned}$$

Inserting e_ζ with ζ as above into this equation gives $\partial_{\bar z} w - (1 - \partial_{\bar z}\bar\psi)\lambda w = 0$, which proves that (a)(i) implies (a)(ii).

On the other hand, if (a)(ii) is satisfied, it follows from (9) that for every $f \in C_P^\infty(X)$ we have

$$P(\partial)(C_{w,\psi}(f)) = \partial_{\bar z}\bar\psi \, C_{w,\psi}(P(\partial)f),$$

which proves (a)(i).

To finish the proof of (a), let $Y \subseteq X$ be open and assume that $C_{w,\psi}$ is well-defined on $C_P^\infty(X)$. Using (a)(ii) it is straightforward to derive (compare equation (9))

$$\forall f \in C^\infty(Y): \quad P(\partial)(C_{w,\psi}(f)) = \partial_{\bar z}\bar\psi \, C_{w,\psi}(P(\partial)f).$$

In order to prove (b), we first notice that for $f \in C^\infty(X)$,

$$\begin{aligned} P(\partial)(w \cdot f(\psi)) = P(\partial)w \cdot f(\psi) &+ \sum_{l=1}^{d}(2\langle \nabla w, \nabla\psi_l\rangle + w\Delta\psi_l)\partial_l f(\psi) \\ &+ w\Big(\sum_{l=1}^{d}\sum_{m=1}^{d}(\partial_l\partial_m f)(\psi)\langle\nabla\psi_l, \nabla\psi_m\rangle\Big). \end{aligned} \tag{14}$$

We first show that (b)(i) implies (b)(ii). Inserting $f = e_{\zeta_j}, 1 \leq j \leq d$, with $\zeta_j = \sqrt{\lambda}\,(\delta_{j,l})_{1\leq l\leq d}$ for any root $\sqrt{\lambda}$ of λ it follows from $e_{\zeta_j} \in C_P^\infty(X)$ that

$$0 = \big(P(\partial)w + \sqrt{\lambda}\,(2\langle\nabla w, \nabla\psi_j\rangle + w\Delta\psi_j) + \lambda w|\nabla\psi_j|^2\big)e_{\zeta_j},$$

so that for every $1 \leq j \leq d$,

$$(15) \qquad 0 = P(\partial)w + \sqrt{\lambda}\,(2\langle \nabla w, \nabla\psi_j\rangle + w\Delta\psi_j) + \lambda w|\nabla\psi_j|^2$$
$$= \big(\Delta w - (1-|\nabla\psi_j|^2)\lambda w\big) + \sqrt{\lambda}\,(2\langle \nabla w, \nabla\psi_j\rangle + w\Delta\psi_j).$$

Analogously, inserting $f = e_{\tilde{\zeta}_j}$, $1 \leq j \leq d$, with $\tilde{\zeta}_j = -\sqrt{\lambda}\,(\delta_{j,k})_{1\leq k\leq d}$ into (14) yields, for $1 \leq j \leq d$,

$$(16) \qquad 0 = \big(\Delta w - (1-|\nabla\psi_j|^2)\lambda w\big) - \sqrt{\lambda}\,(2\langle \nabla w, \nabla\psi_j\rangle + w\Delta\psi_j).$$

Adding (15) and (16) gives

$$(17) \qquad 0 = \Delta w - (1-|\nabla\psi_j|^2)\lambda w = P(\partial)w + \lambda|\nabla\psi_j|^2 w$$

for every $1 \leq j \leq d$, while subtracting (16) from (15) gives

$$(18) \qquad \forall 1 \leq j \leq d: \quad 0 = 2\langle \nabla w, \nabla\psi_j\rangle + w\Delta\psi_j$$

for $\lambda \neq 0$. If $\lambda = 0$ we have $f(x) = x_j \in C_P^\infty(X)$, and plugging this into (14) shows that (18) is also valid for $\lambda = 0$.

Next we insert $f = e_{\eta^\pm_{j,k}}$, $1 \leq j \neq k \leq d$, into (14), where for $\alpha \in \mathbb{C}\setminus\{0,-\lambda\}$, $\eta^\pm_{j,k} = \sqrt{\lambda+\alpha}\,(\delta_{j,l})_{1\leq l\leq d} \pm i\sqrt{\alpha}(\delta_{k,l})_{1\leq l\leq d}$, resulting in

$$(19) \qquad 0 = P(\partial)w + \sqrt{\lambda+\alpha}\,(2\langle \nabla w, \nabla\psi_j\rangle + w\Delta\psi_j)$$
$$\pm i\sqrt{\alpha}\,(2\langle \nabla w, \nabla\psi_k\rangle + w\Delta\psi_k)$$
$$+ w\big((\lambda+\alpha)|\nabla\psi_j|^2 - \alpha|\psi_k|^2 \pm 2i\sqrt{\alpha}\sqrt{\lambda+\alpha}\,\langle \nabla\psi_j, \nabla\psi_k\rangle\big).$$

Subtracting from the "+" version of the above equation the "−" version yields

$$0 = 2i\sqrt{\alpha}\,(2\langle \nabla w, \nabla\psi_k\rangle + w\Delta\psi_k) + w4i\sqrt{\alpha}\sqrt{\lambda+\alpha}\,\langle \nabla\psi_j, \nabla\psi_k\rangle,$$

so taking into account (18) we derive

$$(20) \qquad \forall 1 \leq j \neq k \leq d: \quad 0 = w\langle \nabla\psi_j, \nabla\psi_k\rangle.$$

On account of (18) and (20), equation (19) combined with (17) gives

$$(21) \qquad 0 = \Delta w - (1-|\nabla\psi_j|^2)\lambda w + \alpha w(|\nabla\psi_j|^2 - |\nabla\psi_k|^2)$$
$$= \alpha w(|\nabla\psi_j|^2 - |\nabla\psi_k|^2).$$

Since $\alpha \neq 0$, equation (21) together with (18) and (20) now gives (b)(ii).

Conversely, if (b)(ii) is satisfied, then (14) simplifies to

$$\forall f \in C^\infty(X): \quad P(\partial)(w\cdot f(\psi)) = -\lambda w|\nabla\psi_1|^2 \cdot f(\psi) + w|\nabla\psi_1|^2(\Delta f)(\psi)$$
$$= |\nabla\psi_1|^2 w(P(\partial)f)(\psi),$$

in particular $C_{w,\psi}$ is well-defined on $C_P^\infty(X)$, proving (b)(i).

To finish the proof of (b), let $Y \subseteq X$ be open and assume that $C_{w,\psi}$ is well-defined on $C_P^\infty(X)$. Using (b)(ii) it is straightforward to derive (compare (14))

$$\forall f \in C^\infty(Y): \quad P(\partial)(C_{w,\psi}(f)) = |\nabla\psi_1|^2\, C_{w,\psi}(P(\partial)f). \ \blacksquare$$

Corollary 6.7. *Let $d \geq 2$, $\lambda \in \mathbb{C}$ and $P(\xi) = \sum_{j=1}^d \xi_j^2 - \lambda$. Moreover, let $X \subseteq \mathbb{R}^d$ be homeomorphic to $\mathbb{R}^d$ and assume that smooth mappings $w : X \to \mathbb{C}$ and $\psi : X \to X$ are such that $C_{w,\psi}$ is well-defined on $C_P^\infty(X)$.*

(a) *For $C_{w,\psi}$ the following are equivalent:*
   (i) *$C_{w,\psi}$ is weakly mixing on $C_P^\infty(X)$.*
   (ii) *w has no zeros, ψ is injective and run-away, and* $\det J\psi(x) \neq 0$ *for each $x \in X$.*

   If additionally $|w(x)| \leq 1$ for all $x \in X$, the above are equivalent to

   (iii) *$C_{w,\psi}$ is hypercyclic on $C_P^\infty(X)$.*

(b) *For $C_{w,\psi}$ the following are equivalent:*
   (i) *$C_{w,\psi}$ is mixing on $C_P^\infty(X)$.*
   (ii) *w has no zeros, ψ is injective and strong run-away, and* $\det J\psi(x) \neq 0$ *for each $x \in X$.*

Proof. Because $C_{w,\psi}$ is well-defined on $C_P^\infty(X)$ it follows from Proposition 6.6(b) that for all $Y \subseteq X$ open we have

$$\forall f \in C^\infty(Y): \quad P(\partial)(C_{w,\psi}(f)) = |\nabla\psi_1|^2\, C_{w,\psi}(P(\partial)f). \tag{22}$$

Clearly, (22) implies that $C_{w,\psi}$ acts locally on $C_{w,\psi}(X)$ so that by Theorem 6.2 we only have to show that (a)(ii) implies (a)(i) and that (b)(ii) implies (b)(i). This will be done once we have shown that under (a)(ii), respectively (b)(ii), for each $m \in \mathbb{N}_0$ and every open, relatively compact $Y \subseteq \psi^m(X)$ we have

$$r_X^{(\psi^m)^{-1}(Y)}(C_P^\infty(X)) \subseteq \overline{(C_{w,\psi,(\psi^{m-1})^{-1}(Y)} \circ \cdots \circ C_{w,\psi,Y})(C_P^\infty(Y))}.$$

If (a)(ii), respectively (b)(ii), holds we have $\det J\psi(x) \neq 0$ for each $x \in X$ and we conclude that $\psi^m(X)$ is open and $(\psi^m)^{-1} : \psi^m(X) \to X$ is smooth for every $m \in \mathbb{N}$ and $|\nabla\psi_1(x)|^2 \neq 0$ for all $x \in X$. Moreover, (22) implies for every open set $Y \subseteq \psi^m(X)$ and every $f \in C^\infty((\psi^m)^{-1}(Y))$ that

$$P(\partial)f = |\nabla\psi_1|^{2m} \prod_{j=0}^{m-1} w(\psi^j(\cdot))\Bigg(\bigg(P(\partial)\bigg[\bigg(\frac{f}{\prod_{j=0}^{m-1} w(\psi^j(\cdot))}\bigg)\circ(\psi^m)^{-1}\bigg]\bigg)\circ\psi^m\Bigg),$$

which in turn yields

$$P(\partial)\bigg(\bigg(\frac{f}{\prod_{j=0}^{m-1} w(\psi^j(\cdot))}\bigg)\circ(\psi^m)^{-1}\bigg) = \bigg(\frac{P(\partial)f}{|\nabla\psi_1|^{2m}\prod_{j=0}^{m-1} w(\psi^j(\cdot))}\bigg)\circ(\psi^m)^{-1}.$$

Therefore, for each $m \in \mathbb{N}$, for every open $Y \subseteq \psi^m(X)$, and every $f \in C_P^\infty((\psi^m)^{-1}(Y))$ it follows that

$$\bigg(\frac{f}{\prod_{j=0}^{m-1} w(\psi^j(\cdot))}\bigg)\circ(\psi^m)^{-1} \in C_P^\infty(Y),$$

so that by Remark 3.10(iii) we have

$$C_P^\infty((\psi^m)^{-1}(Y)) \subseteq (C_{w,\psi,(\psi^{m-1})^{-1}(Y)} \circ \cdots \circ C_{w,\psi,Y})(C_P^\infty(Y)).$$

In particular,

$$r_X^{(\psi^m)^{-1}(Y)}(C_P^\infty(X)) \subseteq \overline{(C_{w,\psi,(\psi^{m-1})^{-1}(Y)} \circ \cdots \circ C_{w,\psi,Y})(C_P^\infty(Y))},$$

which is all that had to be shown. ∎

We close this section by applying Theorem 6.2 to characterize the dynamics of weighted composition operators on eigenspaces of the Cauchy–Riemann operator. For $\lambda = 0$ the equivalence of (iii) and (iv) is established in [YoRe], while the equivalence of (iii) and (i) for $\lambda = 0$ was also considered in [Bes], both without any restriction on the range of the weight w.

Corollary 6.8. *Let $d = 2$, $\lambda \in \mathbb{C}$ and $P(\xi) = \frac{1}{2}(\xi_1 + i\xi_2) - \lambda$. Moreover, let $X \subseteq \mathbb{R}^2$ be homeomorphic to $\mathbb{R}^2$ and assume that $w : X \to \mathbb{C}$ and $\psi : X \to X$ are smooth and $C_{w,\psi}$ is well-defined on $C_P^\infty(X)$. Then the following are equivalent:*

(i) *$C_{w,\psi}$ is mixing on $C_P^\infty(X)$*
(ii) *$C_{w,\psi}$ is weakly mixing on $C_P^\infty(X)$.*
(iii) *w has no zeros, ψ is injective, holomorphic, and has no fixed point.*

If additionally $|w(x)| \leq 1$ for all $x \in X$ then the above are equivalent to

(iv) *$C_{w,\psi}$ is hypercyclic on $C_P^\infty(X)$.*

Proof. We identify $\mathbb{R}^2$ with $\mathbb{C}$. Because $C_{w,\psi}$ is well-defined on $C_P^\infty(X)$ it follows from Proposition 6.6(b) that for all $Y \subseteq X$ open we have

$$\forall f \in C^\infty(Y): \quad P(\partial)(C_{w,\psi}(f)) = \partial_{\bar{z}}\bar{\psi}\, C_{w,\psi}(P(\partial)f). \tag{23}$$

This implies that $C_{w,\psi}$ acts locally on $C_{w,\psi}(X)$.

Clearly, (i) implies (ii), and by Theorem 6.2, if (ii) holds, then in particular w has no zeros, ψ is injective and run-away, so ψ has no fixed point. Since w has no zeros it follows from Proposition 6.6(a)(ii) that ψ is holomorphic, so that (ii) implies (iii).

Next, if (iii) holds, the composition operator C_ψ is in particular a well-defined continuous linear operator on $\mathscr{H}(X)$, the holomorphic functions on X. Since X is homeomorphic to $\mathbb{C}$, X is a simply connected domain in $\mathbb{C}$, and because ψ is injective and has no fixed point, it follows from [Bes, proof of Theorem 3.1] that ψ is strong run-away. From the injectivity of the holomorphic mapping ψ we conclude that $0 \neq |\psi'(z)| = \det J\psi(z)$ for all $z \in X$ (see e.g [Ru, Theorem 10.33]). In view of Theorem 6.2 we only have to show that for each $m \in \mathbb{N}_0$ and every open, relatively compact $Y \subseteq \psi^m(X)$,

$$r_X^{(\psi^m)^{-1}(Y)}(C_P^\infty(X)) \subseteq \overline{(C_{w,\psi,(\psi^{m-1})^{-1}(Y)} \circ \cdots \circ C_{w,\psi,Y})(C_P^\infty(Y))}.$$

From $0 \neq \det J\psi(x) = |\partial_z \psi(x)|^2$ for all $x \in X$ together with $\partial_{\bar{z}} \bar{\psi} = \overline{\partial_z \psi}$ it follows that $\partial_{\bar{z}} \bar{\psi}$ has no zeros in X. Thus, (23) can be used just as (22) in the proof of Corollary 6.7 to prove that (i) holds.

Finally, if $|w(x)| \leq 1$ for all $x \in X$ it follows from Theorem 6.2 that (ii) and (iv) are equivalent, which completes the proof. ∎

Motivated by the previous result we close this paper with two open problems. While the first one is concerned with the general abstract setting, the second one aims at a more manageable characterization of hypercyclicity/mixing for weighted composition operators on eigenspaces of the Laplace operator.

PROBLEM 6.9. *1. Are the additional assumptions in Proposition* 3.4(v) *superfluous to prove that ψ is run-away whenever $C_{w,\psi}$ is transitive? If this is the case, the additional assumption on w (or the sheaf $\mathscr{F}$) in Theorem* 3.9 *can be removed so that* (i)–(iv) *in Theorem* 6.2 *are equivalent as also are* (i)–(iii) *in Corollary* 6.7(a) *and* (i)–(iv) *in Corollary* 6.8, *without the additional assumption on the range of the weight w.*

2. Let $X \subseteq \mathbb{R}^d$ be homeomorphic to $\mathbb{R}^d$, $\lambda \in \mathbb{C}$, and $P(\xi) = \sum_{j=1}^d \xi_j^2 - \lambda$. Characterize those $w \in C^\infty(X)$, $|w| \leq 1$, and smooth $\psi : X \to X$ such that

(i) $\forall 1 \leq j \neq k \leq d : |\nabla\psi_j|^2 = |\nabla\psi_k|^2$ *and* $\langle \nabla\psi_j, \nabla\psi_k \rangle = 0$,
(ii) $\forall 1 \leq j \leq d : w\Delta\psi_j + 2\langle \nabla w, \nabla\psi_j \rangle = 0$ *and* $\Delta w - \lambda w = -\lambda|\nabla\psi_1|^2$,
(iii) *ψ is run-away.*

Are there hypercyclic weighted composition operators on $C_P^\infty(X)$ which are not mixing?

Acknowledgements. The author would like to thank J. Wengenroth for pointing out Alexander Duality [Ha, Theorem 3.44] in connection with the version of the Jordan–Brouwer Separation Theorem which is used in this article. Moreover, the author is indebted to one of the anonymous referees for pointing out a gap in the proof of the previous version of Proposition 3.4(v).

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Thomas Kalmes
Faculty of Mathematics
Chemnitz Technical University
09107 Chemnitz, Germany
E-mail: thomas.kalmes@math.tu-chemnitz.de